MICROSCOPIC DERIVATION OF THE ADIABATIC GAS LAW $VT^{f/2} = \text{CONSTANT}$

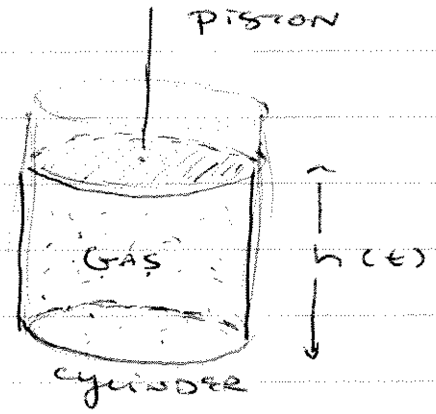
MACROSCOPICALLY IT IS INTUITIVELY CLEAR THAT IF WE COMPRESS A GAS FAST ENOUGH SO THAT NO HEAT ESCAPES* WE WILL INCREASE THE TEMPERATURE OF THE GAS.

MICROSCOPICALLY, HOWEVER, IT IS LESS CLEAR JUST HOW ENERGY IS GETTING TRANSFERRED INTO THE GAS. IN THIS PROBLEM WE LOOK AT HOW THIS OCCURS BY WORKING THROUGH A MICROSCOPIC (i.e. "KINETIC") ~~MODEL~~ DERIVATION OF ADIABATIC COMPRESSION / EXPANSION LAW, $VT^{f/2} = \text{CONSTANT}$

* i.e. "ADIABATICALLY".

CONSIDER A PISTON COMPRESSING / EXPANDING

A CYLINDER OF GAS :



THE HEIGHT OF THE CYLINDER AT A TIME t IS GIVEN BY $h(t)$.

Q1: SHOW THAT THE ADIABATIC LAW

$$V T^{f/2} = \text{CONSTANT}$$

CAN BE WRITTEN AS

$$\frac{d}{dt} h T^{f/2} = 0$$

Q2: SHOW THAT THE ADIABATIC LAW IS EQUIVALENT TO THE EQUATION

$$\dot{n}T + \frac{f}{2} n \dot{T} = 0$$

WHERE $\dot{g}$ DENOTES THE TIME DERIVATIVE OF g (i.e. $\dot{g} \equiv \frac{d}{dt} g$)

Q3: NEXT SHOW THAT THE ADIABATIC LAW IS EQUIVALENT TO:

$$\dot{n}\left[\frac{1}{2} m \overline{v^2}\right] + \frac{n}{2} \frac{d}{dt}\left(f \cdot \frac{1}{2} m \overline{v^2}\right) = 0$$

WHERE m IS THE MASS OF A GAS PARTICLE AND $\overline{v^2}$ IS A PARTICLE'S AVERAGE SQUARED VELOCITY, IN A SINGLE DIRECTION.

THE NEXT STEP IS NOT SO OBVIOUS.

HERE WE ASSUME THE COMPRESSION /

EXPANSION IS SLOW ENOUGH THAT THE GAS MOLECULES CAN REDISTRIBUTE THEIR ENERGY SO THAT AT ALL TIMES THERE IS THE SAME AVERAGE ENERGY IN ~~A~~ TRANSLATIONAL KINETIC ENERGY IN SOME DIRECTION AS THERE IS IN ANY OTHER [PRESUMABLY QUADRATIC] DEGREE OF FREEDOM.

FROM THIS WE GET

$$f \cdot \frac{1}{2} m \overline{v^2} = \overline{\varepsilon}$$

WHERE $\overline{\varepsilon}$ IS THE AVERAGE ENERGY OF A ~~GIVEN~~ GAS MOLECULE.

THE TERM $\frac{d}{dt}\overline{\varepsilon}$ IS THEN THE TIME RATE OF CHANGE OF THE AVERAGE MOLECULAR ENERGY.

Q4: EXPLAIN THE REASONING BEHIND THE ASSOCIATION

$$\frac{d}{dt}\bar{E} = \overline{\Gamma \, \Delta E}$$

WHERE Γ IS THE RATE A WHICH A MOLECULE W/ A VELOCITY COMPONENT v IN THE DIRECTION PERPENDICULAR TO THE PISTON COLLIDES W/ THE PISTON, AND ΔE IS THE ENERGY TRANSFERRED ~~BETWEEN~~ TO THE MOLECULE ~~AND~~ PER COLLISION.

FROM SCHROEDER EQ^N (1.10) WE GET $\Gamma = v/2h$, BUT WHAT ABOUT ΔE?

Q5: SOLVE FOR ΔE IN THE LIMIT $\dot{h} \ll v$.

IS THIS ~~limit~~ APPROXIMATION JUSTIFIED FOR

REALISTIC SCENARIOS?

Spoiler
¡WARNING! HINT: ~~DO AS GALILEO WOULD!~~ ASSUME THE MOLECULES

COLLIDE ELASTICALLY WITH THE PISTON IN THE REFERENCE

FRAME MOVING WITH THE PISTON, i.e.

WHAT WOULD GALILEO DO?

Q6: TAKE THE AVERAGE $\overline{F\Delta E}$ AND COMPLETE

THE DERIVATION.

ANSWERS:

Q1: THE CROSS SECTIONAL AREA A OF THE PISTON IS CONSTANT SO IF

$$V T^{f/2} = \text{CONSTANT} \equiv c$$

THEN $$h T^{f/2} = \frac{V}{A} T^{f/2} = \frac{c}{A}$$

WHICH IS ALSO A CONSTANT. SAME HOLDS FOR OTHER DIRECTION SO THAT IF $h T^{f/2}$ = CONSTANT THEN SO ~~DOES~~ IS $V T^{f/2}$, SO THE TWO CONDITIONS ARE EQUIVALENT.

IF $h T^{f/2}$ = CONSTANT, THEN

$$\frac{d}{dt}\left[h T^{f/2}\right] = \frac{d}{dt}\left[\text{CONSTANT}\right] = 0$$

AND VICE VERSA SO THAT IF $\frac{d}{dt}\left(h T^{f/2}\right) = 0$

THEN $$h(t_2) T^{f/2}(t_2) = h(t_1) T^{f/2}(t_1) + \int_{t_1}^{t_2} dt \, \frac{d}{dt}\left[h T^{f/2}\right] = h(t_1) T^{f/2}(t_1)$$

$\longrightarrow$ CONSTANT

(PRODUCT RULE)

Q2: $\frac{d}{dt} \, h T^{f/2} = \dot{h} T^{f/2} + h\,(f/2)\,T^{f/2 - 1}\,\dot{T}$

$= T^{f/2-1}\left[\dot{h}T + \frac{f}{2} h \dot{T}\right] = 0$

SINCE $T > 0$ THIS IS EQUIVALENT

TO THE CONDITION

$$\dot{h}T + \frac{f}{2} h \dot{T} = 0$$

Q3: MULTIPLY BY $\frac{1}{2}k$ AND APPLY

EQ^N (1.15) FROM SCHROEDER.

Q4: $\Gamma \Delta \varepsilon$ IS THE RATE OF ENERGY

INCREASE ~~FOR~~ DUE TO MOLECULES OF

VELOCITY v $\perp$ TO THE PISTON.

SINCE THE MOLECULES CONTINUALLY

REDISTRIBUTE THEIR ENERGY AMONGST ~~EACH~~

~~OTHER~~ THEMSELVES, WE HAVE

$E = N\bar{\varepsilon}$ AND $\frac{d}{dt} E = N\,\overline{\Gamma \Delta \varepsilon}$ ...

Q5: IN A FRAME MOVING W/ THE PISTON ~~#~~ THE MOLECULE NOW HAS A INCIDENT VELOCITY OF $v - \dot{h}$ AND THUS AN EXIT ~~SPEED~~ VELOCITY OF $-(v-\dot{h})$ IN THE MOVING FRAME.

BACK IN THE LAB FRAME, HOWEVER, THE PARTICLE HAS AN EXIT VELOCITY OF $-(v-\dot{h}) + \dot{h} = 2\dot{h} - v$.

THE MOLECULE THEREFORE EXPERIENCES A CHANGE IN ENERGY OF

$$\frac{1}{2}m(2\dot{h}-v)^2 - \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left[v^2 - 4v\dot{h} + 4\dot{h}^2 - v^2\right]$$

$$= \frac{1}{2}m\left[4\dot{h}^2 - 4v\dot{h}\right]$$

$$\simeq -2mv\dot{h} \quad \left(\dot{h} \ll v\right)$$

Q6: $\dot{h}\left[\frac{1}{2}m\overline{v^2}\right] + \frac{h}{2}\overline{r\Delta\varepsilon}$

$$= \dot{h}\left[\frac{1}{2}m\overline{v^2}\right] + \frac{h}{2}\cdot\overline{\frac{v}{2h}(-2mv\dot{h})}$$

$$= \dot{h}\left[\frac{1}{2}m\overline{v^2}\right] - \dot{h}\left[\frac{1}{2}m\overline{v^2}\right]$$

$$= 0 \quad \checkmark$$

NOTICE I WAS CAREFUL HERE NOT TO CONFUSE $\overline{v^2}$ W/ $\bar{v}\,\bar{v} = \bar{v}^2$.