HW Solutions # 4

2.8) a) 21, corresponding to the # of ways of splitting up 20 units of energy between two systems

$\underset{A}{0}, \underset{B}{20}; \underset{A}{1}, \underset{B}{19}; \cdots ; \underset{A}{20}, \underset{B}{0}$

1 2 ... 21

b) $\Omega(20,20) = \binom{20+20-1}{20} = \binom{39}{20}$

~~(2.21) $\approx \left(\frac{e \cdot 20}{20}\right)^{20} = e^{20} \approx 10^{20/2.3}$~~

~~$\frac{1}{2}$ billion~~ $\approx$ 69 billion

This is the ways of dividing out 20 energy units amongst the combined 20 oscillators.

c) $\Omega_{total} = \Omega_A \Omega_B = \cancel{\binom{20-1+20}{20}\binom{20-1}{0}}$

$= \binom{20+10-1}{20}\binom{10-1}{0}$

$= \binom{29}{20}\binom{9}{0} = \binom{29}{20} \cdot 1$

$\approx$ 10 million

d) $\binom{\overset{N_A}{10} + \overset{q_A}{10} - 1}{\underset{10}{q_A}}\binom{\overset{N_B}{10} + \overset{q_B}{10} - 1}{\underset{10}{q_B}} = \binom{19}{10}^2 \approx 8.5$ billion

e) IF WE INITIALIZE THE COMBINED SYSTEM W/ A LARGE IMBALANCE IN ENERGY QUANTA BETWEEN SYSTEMS A + B, THEN THE SYSTEM WILL IRREVERSIBLY TRANSFER QUANTA TO EVEN ~~OUT~~ IT OUT, i.e. MAXIMIZE PROBABILITY.

2) ACCORDING TO THE "STATISTICAL 2ND LAW OF T.D.", A SYSTEM WILL BE IN EQUILIBRIUM WHEN ITS MACROSTATE IS THE ONE W/ THE LARGEST NUMBER OF MICROSTATES.

FOR A SYSTEM W/ TWO SUBSYSTEMS A + B IN THERMAL CONTACT [i.e. CAN EXCHANGE ENERGY], THE ~~[illegible]~~ NUMBER OF MICROSTATES IS THE PRODUCT OF THEIR INDIVIDUAL MICROSTATES:

$$\Omega_{TOTAL} = \Omega_A \cdot \Omega_B$$

IF WE TRANSFER A SMALL AMOUNT OF ENERGY dU ~~FROM~~ TO SYSTEM A ~~TO~~ FROM SYSTEM B, THEN THE CHANGE IN THE TOTAL NUMBER OF STATES IS:

$$\Omega_A(U_A + dU)\,\Omega_B(U_B - dU)$$

$$= \Big[\Omega_A(U_A) + \underbrace{\frac{d\Omega_A}{dU_A}}_{\equiv \Omega_A'}(U_A)\,dU\Big]$$

$$\cdot \Big[\Omega_B(U_B) + \underbrace{\frac{d\Omega_B}{dU_B}}_{\equiv \Omega_B'}(U_B)(-dU)\Big]$$

in limit as $dU \to 0$:

$$= \Omega_A(U_A)\,\Omega_B(U_B) + \big[\Omega_A'\,\Omega_B - \Omega_A\,\Omega_B'\big]\,dU$$

SO THE CHANGE IN THE TOTAL # OF MICROSTATES is:

$$\left[\Omega_A'\Omega_B - \Omega_A\Omega_B'\right]du$$

$$= \left[\frac{\Omega_A}{\Omega_A}\Omega_A'\Omega_B - \frac{\Omega_B}{\Omega_B}\Omega_A\Omega_B'\right]du$$

$$= \Omega_A\Omega_B\left(\frac{d}{du_A}\ln\Omega_A - \frac{d}{du_B}\ln\Omega_B\right)du$$

SO THERE IS AN EXTREMUM IS Ω_{TOTAL}

WHEN

$$\frac{d}{du_A}\ln\Omega_A = \frac{d}{du_B}\ln\Omega_B$$

TWO SYSTEMS IN THERMAL CONTACT WILL EXCHANGE ENERGY (HEAT) UNTIL THEIR TEMPERATURES ARE EQUAL, SO WE MIGHT Suspect ~~Expect~~ $\frac{d}{du}\ln\Omega$ TO BE ~~AT LEAST~~

~~IN 1-TO-1 CORRESPONDENCE W/ ITS TEMPERATURE.~~ ~~PROPORTIONAL~~ AT LEAST IN 1-TO-1 CORRESPONDENCE W/ a ~~THE~~ SYSTEM'S TEMPERATURE.

problem 3)
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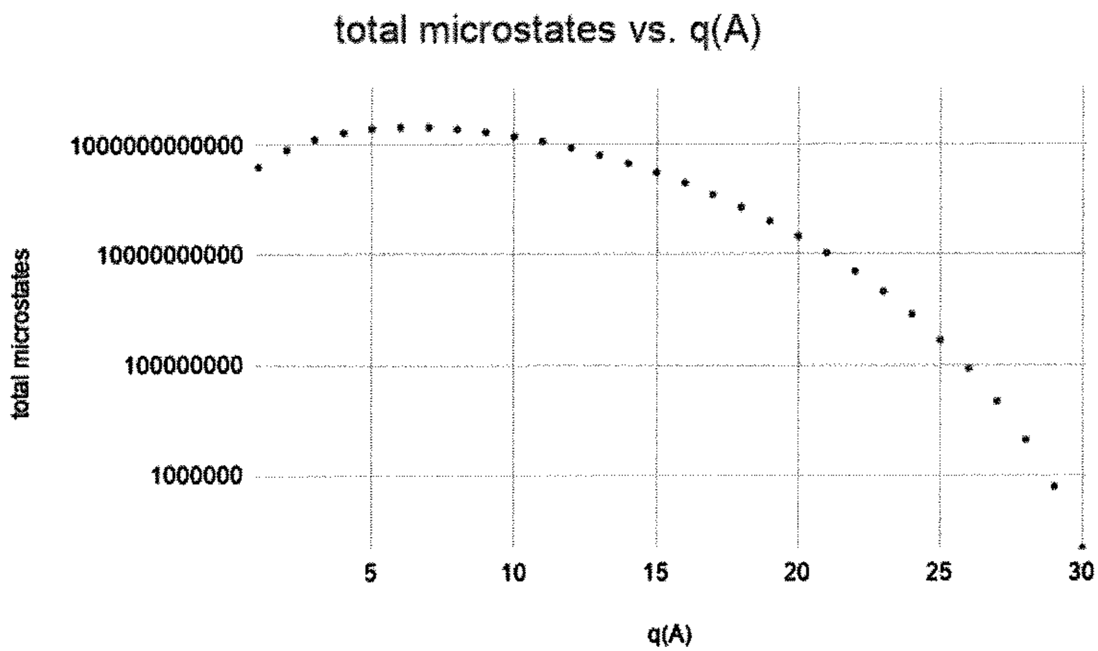
a)

From the attached plot the total number of microstates peaks at q(A) = 6

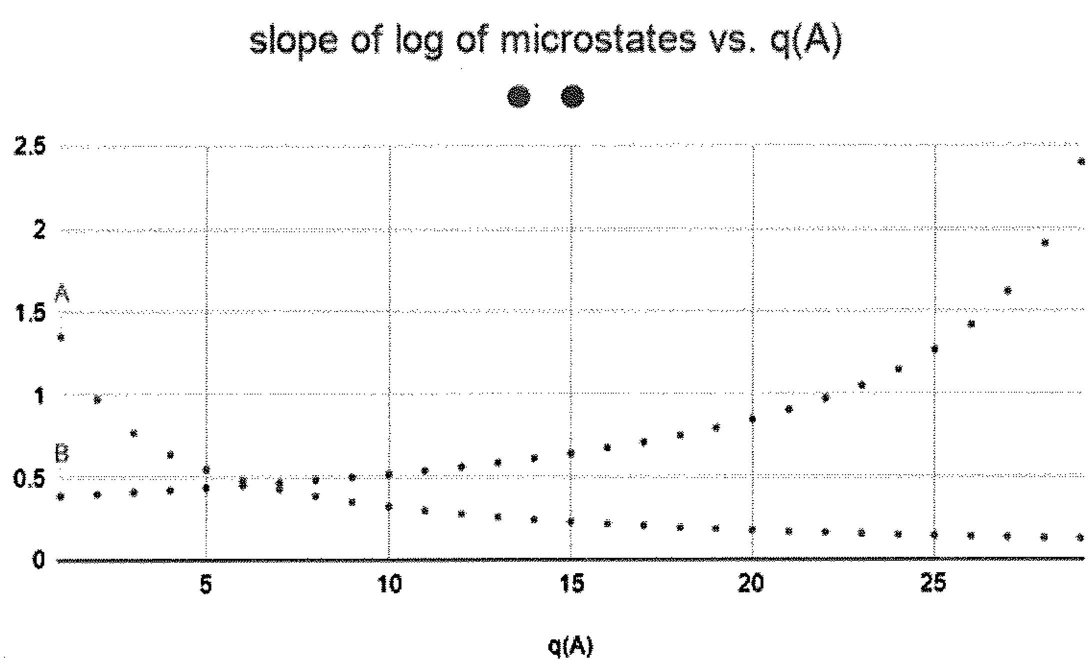
b)

The expression for both systems is a finite analog of a derivative of the log of their number of microstates, so we might expect them to be equal when the system is in equilibrium, i.e. at q_A^MAX. This does appear to be the case.

If we interpret the derivative of the log of the microstates as something analogous to the temperature of the system, then we find that we would expect negligible change in this quantity at q_A^MAX for system A (and thus system B) upon doubling the size of system A. Though doubling the size of A at fixed q will take roughly twice as much heat away from system B, the "temperature" of system B or the slope of its log of number of microstates at equilibrium will only have changed very little because it is so much larger than A. In other words, it has a very large heat capacity.



total microstates vs. q(A)
total microstates
1000000000000
10000000000
100000000
1000000
5
10
15
20
25
30
q(A)



slope of log of microstates vs. q(A)
2.5
2
1.5
1
0.5
0
A
B
5
10
15
20
25
q(A)

4) FROM THE SYMMETRY OF THE RHS OF EQ:(2.17) WE CAN IMMEDIATELY CONVERT THE HIGH TEMP EXPRESSION EQ (2.21) TO LOW TEMP BY EXCHANGING q AND N, GIVING:

$$\Omega(N,q) \simeq \left(\frac{eN}{q}\right)^q \qquad q \ll N$$

FOR TWO SUBSYSTEMS EACH OF SIZE N SPLITTING q ENERGY QUANTA INTO

$$q/2 + \Delta q/2 \quad \text{AND} \quad q/2 - \Delta q/2$$

WE HAVE:

$$\Omega_{TOTAL}(\Delta q) = \left(\frac{eN}{q/2 + \Delta q/2}\right)^{q/2 + \Delta q/2} \left(\frac{eN}{q/2 - \Delta q/2}\right)^{q/2 - \Delta q/2}$$

$$= \left[\left[\frac{2eN}{q}(1+\varepsilon)^{-1}\right]^{1+\varepsilon}\right]^{q/2} \left[\left[\frac{2eN}{q}\right](1-\varepsilon)^{-1}\right]^{1-\varepsilon}\Bigg]^{q/2}$$

$$\varepsilon \equiv \Delta q / q$$

$$= \left[\left[\frac{2Ne}{q}\right]^{1+\varepsilon+1-\varepsilon} \underbrace{(1+\varepsilon)^{-1}(1-\varepsilon)^{-1}}_{(1-\varepsilon^2)^{-1}} \underbrace{(1+\varepsilon)^{-\varepsilon}(1-\varepsilon)^{+\varepsilon}}_{\left[\frac{(1-\varepsilon)}{(1+\varepsilon)}\right]^{\varepsilon}} \right]^{q/2}$$

$$= \left[\frac{2Ne}{q}\right]^{q} \left[\underbrace{(1-\varepsilon^2)^{-1}}_{\simeq 1+\varepsilon^2} \underbrace{\left[\frac{(1-\varepsilon)}{(1+\varepsilon)}\right]^{\varepsilon}}_{e^{\ln\left[\frac{(1-\varepsilon)}{(1+\varepsilon)}\right]^{\varepsilon}}} \right]^{q/2}$$

$$\simeq e^{\varepsilon^2}$$

$$= e^{\varepsilon \ln\left[\frac{(1-\varepsilon)}{(1+\varepsilon)}\right]}$$

$$\simeq e^{\varepsilon \ln[1-2\varepsilon]}$$

$$\simeq e^{\varepsilon(-2\varepsilon)}$$

$$= e^{-2\varepsilon^2}$$

$$\simeq \left(\frac{2Ne}{q}\right)^{q} \left[e^{\varepsilon^2-2\varepsilon^2}\right]^{q/2} = \left(\frac{2Ne}{q}\right)^{q} e^{-\left(\Delta q/\sqrt{q}\right)^2/2}$$

SO THE MICROSTATES ARE GAUSSIAN DISTRIBUTED IN Δq W/ MEAN OF ZERO AND STANDARD DEVIATION OF $\sqrt{q}$