11.3

STARTING W/

(A) $\dot{c}_a = (i\hbar)^{-1} H'_{ab} e^{-i\omega_0 t} c_b$

AND

(B) $\dot{c}_b = (i\hbar)^{-1} H'^{*}_{ab} e^{i\omega_0 t} c_a$

WE TAKE A DER. OF (A) :

$$\ddot{c}_a = (i\hbar)^{-1} H'_{ab} e^{-i\omega_0 t} \left[\dot{c}_b - i\omega_0 c_b \right]$$

THEN SUB IN (A) + (B) TO GET :

$$\ddot{c}_a = -|H'_{ab}|^2/\hbar^2 \, c_a - i\omega_0 \dot{c}_a$$

OR

$$\ddot{c}_a + i\omega_0 \dot{c}_a + |H'_{ab}|^2/\hbar^2 \, c_a = 0$$

LOOK FOR SOLUTIONS OF FORM :

$e^{i\omega t}$:

$$\left[-\omega^2 - \omega\omega_0 + |H'_{ab}|^2/\hbar^2 \right] c_a = 0$$

OR

$$\omega^2 + \omega\omega_0 - |H'_{ab}|^2/\hbar^2 = 0$$

i.e. :

$$\omega_{\pm} = -\bar{\omega} \pm \left[\bar{\omega}^2 + W^2 \right]^{1/2}$$

$$\bar{\omega} \equiv \omega_0/2 \qquad W \equiv |H'_{ab}|/\hbar$$

so

$$C_a(t) = a_+ e^{i\omega_+ t} + a_- e^{i\omega_- t}$$

But $C_a(0) = 1$, $c_b(0) \propto \dot{C}_a(0) = 0$

so $a_+ + a_- = 0$

AND $a_+\omega_+ + a_-\omega_- = 0$

so $a_+(\omega_+ - \omega_-) = \omega_-$

or $a_+ = \omega_- / [\omega_+ - \omega_-]$

so $a_- = 1 - \omega_- / [\omega_+ - \omega_-]$

so
$$C_a(t) = \omega_- / [\omega_+ - \omega_-] \, e^{i\omega_+ t} + \left(1 - \omega_- / [\omega_+ - \omega_-] \right) e^{i\omega_- t}$$

AND
$$c_b(t) = \frac{i}{W} e^{-i\delta} \dot{C}_a(t)$$

where $H'_{ab} = |H'_{ab}| e^{i\delta}$

$$\frac{d}{dt}\left[|C_a|^2 + |C_b|^2 \right]$$

$$= \frac{d}{dt}\left(C_a^* C_a + C_b^* C_b \right]$$

$$= \cancel{\frac{d}{dt}} \; \dot{C}_a^* C_a + C_a^* \dot{C}_a + \dot{C}_b^* C_b + C_b^* \dot{C}_b$$

$$= 2\,\mathrm{Re}\left(\dot{C}_a^* C_a + \dot{C}_b C_b^* \right]$$

$$= 2\,\mathrm{Re}\left[\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} C_b^* C_a \right.$$

$$\left. - \frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} C_a C_b^* \right]$$

$= 0$, so since $|C_a(0)|^2 + |C_b(0)|^2 = 1$

THEN $|C_a(t)|^2 + |C_b(t)|^2 = 1$

FOR ALL TIME

11.6

$$\frac{d}{dt}\left(C_a^{(0)} + \lambda C_a^{(1)} + \lambda^2 C_a^{(2)} + \cdots \right]$$

$$= (i\hbar)^{-1} \lambda H'_{ab} e^{-i\omega_0 t} \left[C_b^{(0)} + \lambda C_b^{(1)} + \lambda^2 C_b^{(2)} + \cdots \right]$$

(AND VICE VERSA], SO TO FIRST ORDER:

$$\dot{C}_a^{(1)} = (i\hbar)^{-1} H'_{ab} e^{-i\omega_0 t} b$$

~~ASSUMING~~ SO

$$C_a^{(1)}(t) = \alpha b \left(e^{-i\omega_0 t} - 1 \right)$$

$$\alpha = H'_{ab} / \hbar\omega_0$$

AND LIKEWISE: $C_b^{(1)}(t) = -\alpha a \left(e^{i\omega_0 t} - 1 \right)$

(ASSUME α IS <u>REAL</u> (WE ALWAYS CAN))

FOR 2^{ND} ORDER:

$$\dot{C}_a^{(2)}(t) = (i\hbar)^{-1} H'_{ab} e^{-i\omega_0 t} c_b^{(1)}(t)$$

(AND VICE VERSA), SO

$$C_a^{(2)}(t) = \cancel{-i\omega_0 t}\, \alpha^2 a \left[1 - e^{-i\omega_0 t} - i\omega_0 t \right)$$

AND $C_b^{(2)}(t) = \alpha^2 b \left(1 - e^{i\omega_0 t} + i\omega_0 t \right]$

11.10

ACCORDING TO THE DISCUSSION IN SECTION 11.3.1, THE SPONTANEOUS EMISSION RATE IS RELATED TO THE STIM. EM. RATE BY:

$$A = B_{ba} \omega_0^3 \hbar / \pi^2 c^3$$

THE STIM EM. RATE IS $B_{ba} \rho(\omega)$

SO WE WOULD LIKE TO INSPECT THE RATIO

$$\frac{\text{Stim}}{\text{Spon}} = \rho(\omega) \Big/ \cancel{B_{ba}} \omega_0^3 \hbar / \pi^2 c^3$$

WHICH, BY 11.59, IS

$$= \frac{\cancel{\hbar}}{\cancel{\pi^2 c^3}} \cancel{\omega^3} / \left(e^{\hbar\omega/k_B T} - 1 \right) \Big/ \cancel{\omega^3} \cancel{\hbar} / \cancel{\pi^2 c^3}$$

$$= \left(e^{\hbar\omega/k_B T} - 1 \right)^{-1}$$

SO STIM EM DOMINATES IF $\hbar\omega \ll k_B T$

OR $\omega \ll \frac{\cancel{1254\ \mu m\ eV}}{8\ \cancel{200\ eV\ nm}\ \mu m} \cdot 3 \cdot \cancel{10^8}\ m/s\ 10^{14}$

$\simeq 3.75 \cdot 10^{13}$ RAD/s

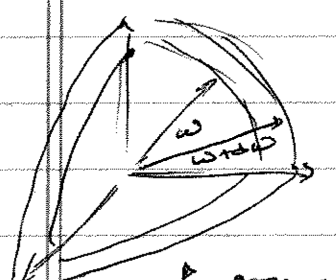
OR $f \lesssim 5 \cdot 10^{12}$ Hz

[VISIBLE LIGHT HAS FREQS IN THE $\underline{100}$'s OF THZ]

11.11

a) We can check easily that the proposed solution satisfies both the diff. E.Q. and the B.C.s. That is enough of a demo for me.

b) By analogy w/ the Fermi gas, we are looking for the product of the energy of modes w/ freq ω and the number of modes w/ freqs between ω & $d\omega + \omega$. Griffiths gives us the former, while the latter is the volume of a shell of inner radius $\frac{l\omega}{\pi c}$ + outer radius



$\frac{l}{\pi c}(\omega + d\omega)$, which is, to first order in $d\omega$:

octant

$$\frac{4\pi}{8}\left[\frac{l}{\pi c}(\omega)\right]^2 \frac{d\omega\, l}{\pi c}$$

Remembering the factor 2 for the polarizations:

$$dE = \frac{2\hbar\omega}{2}\pi\left[\frac{l^2}{\pi^2 c^2}\omega^2\right]\frac{d\omega\, l}{\pi c}$$

or $$\frac{dE}{l^3} = \frac{\hbar\omega^3}{\pi^2 c^3}\, d\omega \quad \text{so} \quad \rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}$$

c) $$R^{spon}_{b\to a} = \frac{\cancel{\pi}}{3\epsilon_0\hbar^{\cancel{2}}}|\wp|^2\frac{\cancel{\hbar}\omega^3}{\pi^{\cancel{2}}c^3} = \frac{|\wp|^2\omega^3}{3\epsilon_0\pi\hbar c^3}$$ ✓