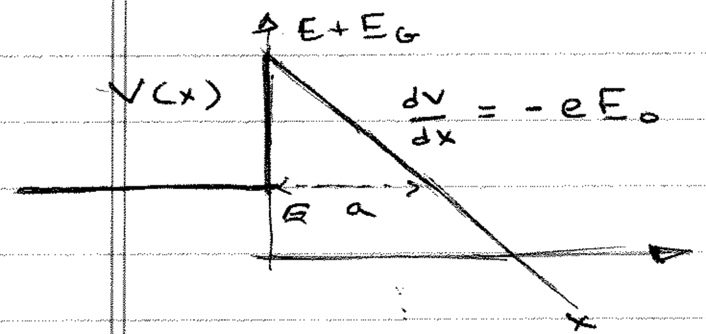
9.5 ZENER TUNNELING

WE MODEL THE POTENTIAL AS FOLLOWS:



$$T \simeq e^{-2\gamma} \qquad \gamma \equiv \frac{1}{\hbar}\int_0^a |p(x)|\,dx$$

[EQ^N 9.23]

HERE $a = E_G / eE_0$

$$p(x) = \left[2m\left(\cancel{E} + eE_0 x - \cancel{E} - E_G\right)\right]^{1/2}$$

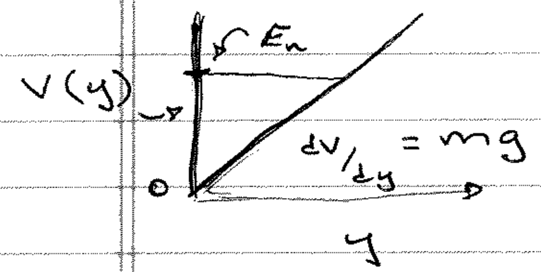
$$= \left[2m\left(E_G\, x/a - E_G\right)\right]^{1/2}$$

$$\gamma = \frac{\left[2mc^2E_G\right]^{1/2}}{\hbar c}\, a \underbrace{\int_0^1 dz\left[-z+1\right]^{1/2}}_{\frac{2}{3}\left[-z+1\right]^{+3/2}\Big|_1^0 \;=\; 2/3}$$

$$= \frac{\sqrt{8}\sqrt{mc^2}\,E_G^{3/2}}{3eE_0\hbar c} \quad \text{OR} \quad \frac{\sqrt{8}\sqrt{m}\,E_G^{3/2}}{3eE_0\hbar}$$

9.7 a)

From 9.48:



$$\int_0^{E_n/mg} dy \sqrt{mg}\sqrt{y}$$

$$\int_0^{E_n/mg} dy \left[E_n - mgy \right]^{1/2}$$

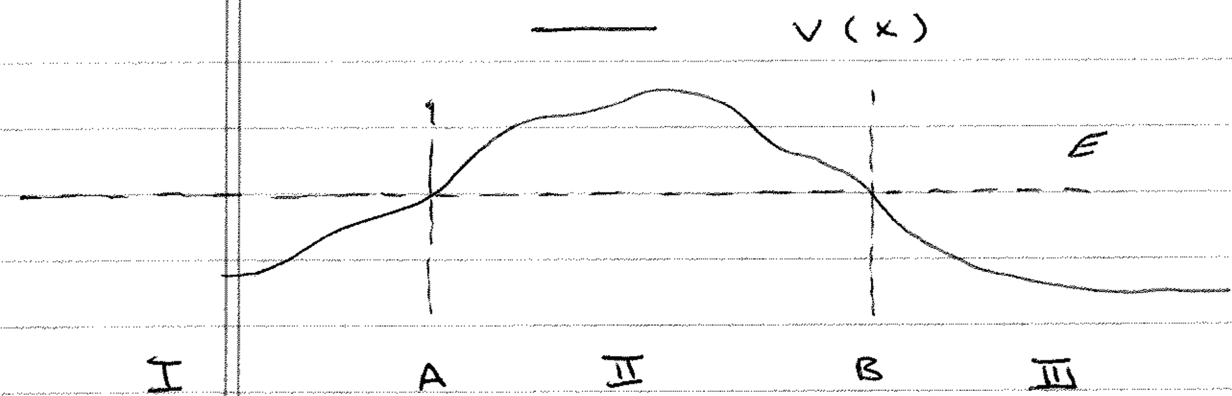
$$= \sqrt{E_n} \int_0^1 dz \left(1 - z \right]^{1/2}$$

$$= \sqrt{E_n}\ \frac{2}{3} \left(1 - z \right]^{+3/2} \Big|_1^0$$

$$= \sqrt{E_n}\ \frac{2}{3} = \left(n - \frac{1}{4} \right) \pi \hbar$$

$$\Longrightarrow E_n = \left(\frac{3}{2} \left(n - \frac{1}{4} \right) \pi \hbar \right]^2$$

9.11



$$\psi^{I}_{WKB} \text{ @ } A \simeq \frac{1}{\sqrt{\hbar}\,\alpha_1^{3/4}\,(-\tilde{x}_1)^{3/4}}$$

$$\left[A \exp\left(i\,\frac{2}{3}(-\alpha_1\tilde{x}_1)^{3/2}\right) + B \exp\left(-i\,\frac{2}{3}(-\alpha_1\tilde{x}_1)^{3/2}\right)\right]$$

$$\left[\tilde{x}_1 \equiv x - x_1\right], \quad \left[\alpha_1 \equiv \left(\frac{2m}{\hbar^2}V'(x_1)\right)^{1/3}\right]$$

WHILE ψ_p^A LEFT OF x_1 IS:

$$\psi_p^A \simeq \cancel{\frac{1}{\sqrt{\pi}}} \cdot \cancel{\frac{1}{\alpha^{3/4}}}\; \frac{a}{\sqrt{\pi}} \cdot \frac{1}{(-\alpha_1\tilde{x}_1)^{1/4}} \sin\left[\frac{2}{3}(-\alpha_1\tilde{x}_1)^{3/2} + \frac{\pi}{4}\right]$$

$$+ \frac{b}{\sqrt{\pi}} \cdot \frac{1}{(-\alpha_1\tilde{x}_1)^{1/4}} \cos\left[\frac{2}{3}(-\alpha_1\tilde{x}_1)^{3/2} + \frac{\pi}{4}\right]$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{(-\alpha_1)^{1/4}(-\tilde{x}_1)^{1/4}} \cdot \left[e^{i\pi/4}(b - ia)\,e^{i\frac{2}{3}(-\alpha_1\tilde{x}_1)^{3/2}} + e^{-i\pi/4}(b + ia)\,e^{-i\frac{2}{3}(-\alpha_1\tilde{x}_1)^{3/2}}\right]$$

MATCHING COEFFICIENTS WE FIND:

$$\frac{A}{\sqrt{\hbar}} \cdot \frac{1}{\alpha_1^{3/4}} = \frac{1}{2} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{(+\alpha_1)^{1/4}} e^{i\pi/4} (b - ia)$$

$$\frac{B}{\sqrt{\hbar}} \cdot \frac{1}{\alpha_1^{3/4}} = \frac{1}{2} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{(+\alpha_1)^{1/4}} e^{-i\pi/4} (b + ia)$$

SIMILARLY, ψ^{II}_{WKB} @ A IS APPROX:

$$\psi^{\text{II}}_{WKB} \simeq \frac{C}{\sqrt{\hbar}\,\alpha_1^{3/4}\,\hat{x}_1^{1/4}} e^{+\frac{2}{3}(\alpha_1 \hat{x}_1)^{3/2}}$$

$$+ \frac{D}{\sqrt{\hbar}\,\alpha_1^{3/4}\,\hat{x}_1^{1/4}} e^{-\frac{2}{3}(\alpha_1 \hat{x}_1)^{3/2}}$$

WHILE ψ_p^A RIGHT OF A IS:

$$\psi_p^A \simeq \frac{a}{2\sqrt{\pi}\,(\alpha_1 \hat{x}_1)^{1/4}} e^{-\frac{2}{3}(\alpha_1 \hat{x}_1)^{3/2}} + \frac{b}{\sqrt{\pi}\,(\alpha_1 \hat{x}_1)^{1/4}} e^{\frac{2}{3}(\alpha_1 \hat{x}_1)^{3/2}}$$

SO THAT: $$\frac{C}{\sqrt{\hbar}} \cdot \frac{1}{\alpha^{3/4}} = \frac{b}{\sqrt{\pi}} \cdot \frac{1}{\alpha_1^{1/4}}$$

AND $$\frac{D}{\sqrt{\hbar}} \cdot \frac{1}{\alpha^{3/4}} = \frac{a}{2\sqrt{\pi}} \cdot \frac{1}{\alpha_1^{1/4}}$$

TO HANDLE TURNING POINT B WE NOTE:

$$\int_{x_1}^{x} |p(x')|\,dx'$$

$$= \underbrace{\int_{x_1}^{x_2} |p(x')|\,dx'}_{\equiv \gamma\hbar} + \int_{x_2}^{x} |p(x')|\,dx'$$

$$\cong \hbar\gamma + \sqrt{2m}\int_{x_2}^{x} |E - (E + V'(x_2)\overset{(x'-x_2)}{\cancel{x_2}})|^{1/2}\,dx'$$

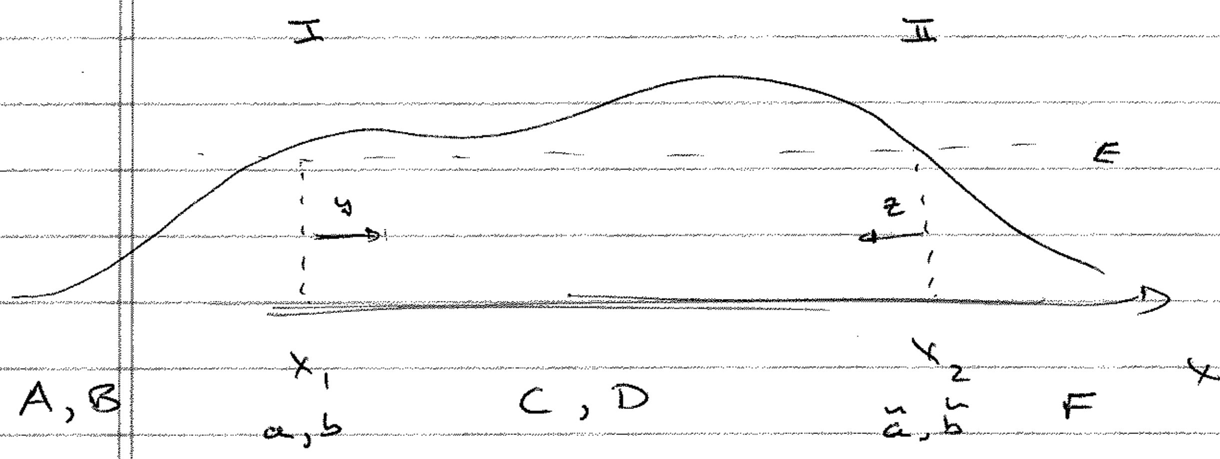
$$\cancel{[\tilde{x}_2 = x' - x_2]}$$

$$= \hbar\gamma + \sqrt{2m}\int_0^{\tilde{x}_2} dx'\,|V'(x_2)\,x'|^{1/2}$$

$$= \hbar\gamma + \sqrt{2m}\,\frac{2}{3}|V'(x_2)|^{1/2}\,(+x')^{3/2}\Big|_0^{\tilde{x}_2}$$

$$= \hbar\left[\gamma + \frac{2}{3}(+\alpha_2\tilde{x}_2)^{3/2}\right]$$

9.11



STRATEGY: WRITE THE
$p(x)$ ~~&~~ $\int_{x_1}^{x_2} p(x)\,dx$
QUANTITIES IN TERMS OF COORDINATES
y NEAR REGION I & z NEAR
REGION II FOR MORE CONVENIENT
COMPARISON TO PATCHING FUNCTIONS.

WE HAVE ONE "KNOWN" QUANTITY A
AND 8 UNKNOWN QUANTITIES
B, C, D, F (WKB COEFFS)
a, b, $\tilde{a}$, $\tilde{b}$ (PATCHING FUNCTION COEFFS)

FOR EACH TRANSITION REGION (I OR II)
~~WE GET~~ AND EACH SIDE OF THE REGION
(LEFT OR RIGHT) WE WILL GET TWO
EQ^{NS} FROM MATCHING COEFFS OF
TWO FUNCTIONS, SO: 8 EQNS, 8 UNKNOWNS.

WKB FUNCTIONS

REGION I :

$$y \equiv x - x_1 \qquad \alpha_1 \equiv \left[\frac{2m}{\hbar^2} V'(x_1)\right]^{1/3}$$

LEFT $(y < 0)$:

$$p(y) = \hbar \alpha_1^{3/2} (-y)^{1/2}$$

$$\int_{y+x_1}^{x_1} dx\, p(y) = \frac{2}{3}\hbar \left[-\alpha_1 y\right]^{3/2}$$

RIGHT $(y > 0)$:

$$|p(y)| = \hbar \alpha_1^{3/2} y^{1/2}$$

$$\int_{x_1}^{x_1+y} |p(\cancel{y})|\,dx = \frac{2}{3}\hbar \left(\alpha_1 y\right]^{3/2}$$

REGION II :

$$z \equiv x_2 - x \qquad \alpha_2 \equiv \left[\frac{2m}{\hbar^2} |V'(x_2)|\right]^{1/3}$$

RIGHT $(z < 0)$

$$p(z) = \hbar \alpha_2^{3/2} (-z)^{1/2}$$

$$\int_{x_2}^{x_2-z} p(x)\,dx = \frac{2}{3}\hbar \left(-\alpha_2 z\right]^{3/2}$$

LEFT $(z > 0)$

$$|p(z)| = \hbar \alpha_2^{3/2} z^{1/2}$$

~~RIGHT~~ ~~(-z)~~

$$\int_{x_1}^{x_2-z} |p(x)|\,dx = I - \frac{2}{3}\hbar \left(\alpha_2 z\right]^{3/2}$$

$$\left(I \equiv \int_{x_1}^{x_2} |p(x)|\,dx\right]$$

SO REGION I LEFT:

$$\psi_{WKB}(y) \simeq \frac{\overset{A}{B}}{\hbar^{1/2}\alpha_1^{3/4}[-y]^{1/4}} \cdot \mathrm{Exp}\left[\mp \frac{i}{\hbar}\frac{2}{3}\hbar(-\alpha_1 y]^{3/2}\right]$$

RIGHT:

$$\psi_{WKB}(y) \simeq \frac{\overset{C}{D}}{\hbar^{1/2}\alpha_1^{3/4}y^{1/4}} \mathrm{Exp}\left[\pm \frac{1}{\hbar}\frac{2}{3}\hbar(\alpha_1 y]^{3/2}\right]$$

REGION II ~~LEFT~~ RIGHT:

$$\psi_{WKB}(z) \simeq \frac{F}{\hbar^{1/2}\alpha_2^{3/4}(-z)^{1/4}} \cdot \mathrm{Exp}\left[\frac{i}{\hbar}\frac{2}{3}\hbar(-\alpha_2 z]^{3/2}\right]$$

LEFT:

$$\psi_{WKB}(z) \simeq \frac{\overset{C}{D}}{\hbar^{1/2}\alpha_2^{3/4}z^{1/4}} \mathrm{Exp}\left[\pm\left[\frac{I}{\hbar} - \frac{1}{\hbar}\frac{2}{3}\hbar(\alpha_2 z]^{3/2}\right]\right]$$

PATCHING FUNCTIONS

REGION I

~~REGION~~

LEFT $(y<0)$

$$\begin{matrix} A_i(y\alpha_1) \\ B_i(y\alpha_1) \end{matrix} \simeq \frac{\overset{a}{b}}{\sqrt{\pi}\,\alpha_1^{1/4}(-y)^{1/4}} \begin{matrix}\sin\\ \cos\end{matrix}\left[\frac{2}{3}(-\alpha_1 y)^{3/2}+\frac{\pi}{4}\right]$$

RIGHT $(y>0)$

$$\begin{matrix} A_i \\ B_i \end{matrix}(y\alpha_1) \simeq \frac{\begin{matrix}(1/2)\\(1)\end{matrix}\ \overset{a}{b}}{\sqrt{\pi}\,\alpha_1^{1/4}\,y^{1/4}} \exp\left[\mp\frac{2}{3}(\alpha_1 y)^{3/2}\right]$$

REGION II

RIGHT $(z<0)$.

SAME AS LEFT SIDE REGION I W/ $y \to z$

$\alpha_1 \to \alpha_2$

AND VICE VERSA.

$a \to \hat{a}$

$b \to \hat{b}$

MATCHING COEFFS

REGION II RIGHT

$$\frac{1}{\sqrt{\pi}\,\alpha_2^{1/4}(-z)^{1/4}}\left[\hat{a}\,\frac{1}{2i}\left(e^{i\left[\frac{2}{3}(-\alpha_2 z)^{3/2}+\pi/4\right]} - e^{-i(\cdots)}\right) + \hat{b}\,\frac{1}{2}\left(e^{i[\cdots]} + e^{-i(\cdots)}\right)\right]$$

$$= \frac{1}{\sqrt{\pi}\,\alpha_2^{1/4}(-z)^{1/4}}\cdot\frac{1}{2}\left[(\hat{b}-i\hat{a})\,e^{i[\cdots]}e^{i\pi/4} + (\hat{b}+i\hat{a})\,e^{-i(\cdots)}e^{-i\pi/4}\right]$$

$$= \frac{F}{\hbar^{1/2}\alpha_2^{3/4}(-z)^{1/4}}\,e^{i(\cdots)}$$

$$\Longrightarrow \frac{F}{\hbar^{1/2}\alpha_2^{3/4}\cancel{(-z)^{1/4}}} = \frac{\hat{b}-i\hat{a}}{\sqrt{\pi}\,\alpha_2^{1/4}}\cdot\frac{1}{2}\,e^{i\pi/4}$$

OR $$F = \left[\frac{\hbar}{\pi}\alpha_2\right]^{1/2}\frac{1}{2}\left[\hat{b}-i\hat{a}\right]e^{i\pi/4}$$

AND $\hat{b}+i\hat{a} = 0$ OR $\hat{a} = i\hat{b}$

SO $$F = \left(\hbar/\pi\,\alpha_2\right]^{1/2}\frac{1}{2}\left[\hat{b}-i(i\hat{b})\right]e^{i\pi/4}$$

$$F = \left(\hbar/\pi\,\alpha_2\right]^{1/2}\hat{b}\,e^{i\pi/4}$$

REGION II LEFT

$$\frac{1}{(\pi)^{1/2} \alpha_2^{1/4} z^{1/4}} \left[\frac{\hat{a}}{2} \exp\left(-(\cdots)\right) + \hat{b} \exp\left(+(\cdots)\right) \right]$$

$$= \frac{1}{\hbar^{1/2} \alpha_2^{3/4} z^{1/4}} \left[C e^{-(\cdots)} e^{+I/\hbar} + D e^{+(\cdots)} e^{-I/\hbar} \right]$$

$$\Longrightarrow \frac{C e^{+I/\hbar}}{\hbar^{1/2} \alpha_2^{3/4} \cancel{z^{1/4}}} = \frac{\hat{a}}{2} \cdot \frac{1}{\pi^{1/2} \alpha_2^{1/4}}$$

$$C = \hat{a} \left(\frac{\hbar \alpha_2}{\pi} \right)^{1/2} \frac{1}{2} e^{-I/\hbar}$$

or $$\hat{a} = 2C \left[\frac{\hbar \alpha_2}{\pi} \right]^{-1/2} e^{+I/\hbar}$$

$$\hat{b} = D \left[\frac{\hbar \alpha_2}{\pi} \right]^{-1/2} e^{-I/\hbar}$$

~~or~~ so $F = D e^{-I/\hbar} e^{i\pi/4}$

REGION I RIGHT :

$$C = b\left[\frac{\hbar\alpha_1}{\pi}\right]^{1/2}$$

$$D = \frac{a}{2}\left[\frac{\hbar\alpha_1}{\pi}\right]^{1/2}$$

REGION I LEFT :

$$b \mp i a = 2\left[\frac{\hbar\alpha_1}{\pi}\right]^{-1/2} e^{\mp i\pi/4} \begin{matrix} A \\ B \end{matrix}$$

so $$F = \frac{a}{2}\left[\frac{\hbar\alpha_1}{\pi}\right]^{1/2} e^{-I/\hbar}\, e^{i\pi/4}$$

$$\cancel{D \pm i a = \mp 2\left[\frac{\hbar\alpha_1}{\pi}\right]^{-1/2} e \left[B e^{+i\pi/4} + A e^{-i\pi/4} \right]}$$

$$b = \left[\frac{\hbar\alpha_1}{\pi}\right]^{-1/2}\left(Be^{+i\pi/4} + Ae^{-i\pi/4}\right)$$

$$a = i\left[\frac{\hbar\alpha_1}{\pi}\right]^{-1/2}\left(Ae^{-i\pi/4} - Be^{i\pi/4}\right)$$

$$b = c\left[\frac{\hbar\alpha_1}{\pi}\right]^{-1/2}$$

$$C = \frac{1}{2}\tilde{a}\left[\frac{\hbar\alpha_2}{\pi}\right]^{1/2} e^{-I/\hbar}$$

$$\tilde{a} = iF\left(\frac{\hbar\alpha_2}{\pi}\right]^{-1/2} e^{-i\pi/4}$$

so $b = \left[\frac{\hbar\alpha_1}{\pi}\right]^{-1/2} \cdot \frac{i}{2} e^{-I/\hbar} e^{-i\pi/4} F$

so $F \cancel{\left[\frac{\hbar\alpha_1}{\pi}\right]^{-1/2}} \frac{i}{2} e^{-I/\hbar} \cancel{e^{-i\pi/4}}$

$$= \cancel{\left[\frac{\hbar\alpha_1}{\pi}\right]^{-1/2}}\left(B\cancel{e^{i\pi/4}}^{\,i\pi/2} + A\cancel{e^{-i\pi/4}}\right)$$

or $e^{-I/\hbar} F = \frac{2}{i}\left(Bi + A\right]$

$$= 2\left(B - iA\right]$$

or $B = F/_2\, e^{-I/\hbar} + iA$

so $F = \frac{i}{2} e^{-I/\hbar} \cancel{e^{i\pi/4}}\left(A\cancel{e^{-i\pi/4}} - B\cancel{e^{i\pi/4}}\, i\right)$

$$= \frac{e^{-I/\hbar}}{2}\left(B + iA\right)$$

so $F = \frac{1}{\cancel{2}} e^{-I/\hbar}\left[\frac{F}{\cancel{2}\;4} e^{-I/\hbar} + iA\cancel{2}\right]$

$$F\left(1-\frac{e^{-2I/\hbar}}{4}\right) = iA\, e^{-I/\hbar}$$

$$F = \frac{iA\, e^{-I/\hbar}}{\left[1-\left[\frac{1}{2}e^{-I/\hbar}\right]^2\right]}$$

$$\left|\frac{F}{A}\right|^2 = \frac{e^{-2I/\hbar}}{\left[1-\left(\frac{1}{2}e^{-I/\hbar}\right]^2\right]^2}$$

For $I \gg \hbar$:

$$\left|\frac{F}{A}\right|^2 \longrightarrow e^{-2I/\hbar}$$