EXCHANGE INTEGRAL

$$\chi(R) = \frac{\alpha}{\pi a^{\cancel{3}2}} \int dr\, r\cancel{{}^2} \frac{e^{-r/a}}{\cancel{r}} \int d\theta \sin\theta\, e^{-r'/a} \underbrace{\int d\phi}_{2\pi}$$

$$z \equiv r/a \qquad z' \equiv r'/a \qquad z_0 \equiv R/a$$

$$= \cancel{\alpha}\, 2 \int dz\, z\, e^{-z} \int d\theta \sin\theta\, e^{-z'}$$

$$z' = z\left[(\beta + \cos\theta)^2 + \sin^2\theta\right]^{1/2}$$

$$\beta \equiv z_0/z$$

$$dz' = \frac{z^2}{z'}\left[-\beta \sin\theta\, d\theta\right]$$

$$\Longrightarrow \sin\theta\, d\theta = -z'dz' \Big/ \beta z^2 = \frac{-z'dz'}{z_0 z}$$

$$\theta = 0 \qquad\qquad \theta = \pi$$

$$z' = |z_0 - z|\,, \quad z' = |z_0 + z| = z_0 + z$$

$$\Longrightarrow \frac{-2}{z_0} \int dz\, e^{-z} \int_{|z_0 - z|}^{z_0 + z} dz'\, e^{-z'} z'$$

$$e^{-z'} z' dz' = -d\left[e^{-z'}(z' + 1)\right]$$

$$\Longrightarrow \frac{2}{z_0}\int_0^\infty dz\, e^{-z}\Big[e^{-[z_0+z]}\,(z_0+z+1)$$

$$- e^{-|z_0-z|}\,(|z_0-z|+1)\Big]$$

$$= \frac{2}{z_0}\Big[e^{-z_0}\int_0^\infty e^{-2z}\,(z_0+z+1)$$

$$- e^{-z_0}\int_0^{z_0} e^{+z-z}\,(z_0-z+1) - e^{+z_0}\int_{z_0}^\infty e^{-z-z}\,(-z_0+z+1)$$

$$\int_a^b e^{-2z}\,dz = -\frac{1}{2}\int_a^b d\,e^{-2z} = \frac{1}{2}\Big[e^{-2a}-e^{-2b}\Big]$$

$$\int_a^b e^{-2z}\,z\,dz = -\frac{1}{2}\int_a^b d\Big[e^{-2z}\,(z+\tfrac{1}{2})\Big]$$

$$= \frac{1}{2}\Big[e^{-2a}\,(a+\tfrac{1}{2}) - e^{-2b}\,(b+\tfrac{1}{2})\Big]$$

$$= \frac{2}{z_0}\Big[e^{-z_0}\Big[(z_0+1)\frac{1}{2}+\frac{1}{2}\Big(\frac{1}{2}\Big)\Big]$$

$$- e^{-z_0}\Big[(z_0+1)\,z_0 - \frac{z_0^2}{2}\Big]$$

$$- e^{+z_0}\Big[(-z_0+1)\Big[\frac{1}{2}e^{-2z_0} - \cancel{\frac{1}{2}}\Big] + \frac{1}{2}e^{-2z_0}\Big(z_0+\frac{1}{2}\Big)\Big]\Big]$$

$$= \frac{\cancel{2}}{\cancel{z_0}} e^{-z_0} \frac{1}{\cancel{2}} \Big(\cancel{z_0} + \cancel{1} + \cancel{\tfrac{1}{2}} - \cancel{2} z_0^{\cancel{2}} - \cancel{2} \cancel{z_0}^{1} + \cancel{z_0}^{2}$$

$$- \cancel{1} + \cancel{z_0} - \cancel{z_0} - \cancel{\tfrac{1}{2}} \Big]$$

$$= -e^{-z_0} \left(z_0 + 1 \right]$$