12.5

a.

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

~~Eigen~~ Eigenvalues are

<u>same</u> as for σ_z, i.e. ± 1

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \cos \theta/2 \\ \sin \theta/2 \, e^{i\phi} \end{bmatrix} = \pm \begin{bmatrix} \cos \theta/2 \\ \sin \theta/2 \, e^{i\phi} \end{bmatrix}$$

$$-i \sin \theta/2 \, e^{i\phi} = \pm \cos \theta/2$$

Solution by inspection is:

$$\theta_\pm = \pi/2 \qquad \phi_\pm = \pm \pi/2$$

So the way we parameterized the state by two <u>real</u> parameters θ + ϕ happens to correspond to arrows pointing along $\pm \hat{y}$. Check that we would get the same for σ_x + σ_z. Why can we describe a two-dimensional state by two <u>real</u> numbers? Shouldn't it require two <u>complex</u> (i.e. <u>four</u> real) numbers? This is called the Bloch <u>sphere</u> representation.

SO:

$$|\pm\rangle_y = \frac{1}{\sqrt{2}}\left[\, |+\rangle_z \pm i\, |-\rangle_z \,\right]$$

THE CORRESPONDING DENSITY OPERATOR IS:

$$|\pm\rangle_y \langle\pm|_y =$$

$$\frac{1}{2}\left[\, |+\rangle_z \pm i\, |-\rangle_z \,\right]\left(\, |+\rangle_z \mp i\, |-\rangle_z \,\right)$$

$$= \frac{1}{2}\Big[\, |+\rangle_z\langle+|_z + |-\rangle_z\langle-|_z$$
$$\pm i\, |-\rangle_z\langle+|_z \mp i\, |+\rangle_z\langle-|_z \,\Big]$$

SO $\rho_{++} = \dfrac{1}{2} = \rho_{--}$

$\rho_{-+} = \pm\dfrac{i}{2} = -\rho_{+-}$

$$\rho^{\pm}_{ij} = \begin{array}{cc} & \begin{matrix} + & \quad - \end{matrix} \\ \begin{matrix} + \\ \\ - \end{matrix} & \begin{bmatrix} \frac{1}{2} & \mp\frac{i}{2} \\ \pm\frac{i}{2} & \frac{1}{2} \end{bmatrix} \end{array} = \frac{1}{2}\left[\, \mathbb{1} \pm \sigma_y \,\right]$$

AND $\rho_\pm^2 = \dfrac{1}{4}\left(\, \mathbb{1} + \overset{\mathbb{1}}{\cancel{\sigma_y^2}} \pm 2\sigma_y \,\right] = \rho_\pm$

12.7

IN THE ~~$|y\rangle$~~ $|\pm\rangle_y$ BASIS THE DENSITY MATRIX WILL NEED TO BE OF THE FORM:

$$\rho_y = \begin{bmatrix} 1/3 & c \\ c^* & 2/3 \end{bmatrix}$$

WHERE c IS A NOT-YET-DETERMINED COMPLEX NUMBER.

IN THE $|\pm\rangle_x$ BASIS WE WILL NEED SOMETHING SIMILAR:

$$\rho_x = \begin{bmatrix} 1/3 & d \\ d^* & 2/3 \end{bmatrix}$$

d IS AGAIN COMPLEX BUT WE DO NOT REQUIRE $d = c$

~~IN THE $|\pm\rangle_x$ BASIS ρ_y WILL BE:~~

CHECK THAT:

$$\rho_y = \frac{1}{2}\mathbb{1} - \frac{1}{6}\sigma_y + \text{Re}(c)\sigma_z - \text{Im}(c)\sigma_x$$

$$\rho^x_{\pm\pm} = \text{Tr}\left(\frac{1}{2}\left[\mathbb{1} \pm \sigma_x\right]\rho_y\right)$$

$$= \mathrm{Tr}\left(\frac{1}{2}\left(\mathbb{1} \pm \sigma_x\right)\right.$$
$$\left.\cdot\left(\frac{1}{2}\mathbb{1} - \frac{1}{6}\sigma_y + \mathrm{Re}(c)\sigma_z - \mathrm{Im}(c)\sigma_x\right)\right)$$

WE USE NOW THE FACT THAT FOR THE PAULI MATRICES:

$$\mathrm{Tr}\,\sigma_i = 0$$
$$\mathrm{Tr}\,\sigma_i\sigma_j = \begin{cases} 0 & i \neq j \\ 2 & i = j \end{cases}$$
(IN FACT $\sigma_i^2 = \mathbb{1}$)

$$= \mathrm{Tr}\left(\frac{1}{4}\mathbb{1} \pm \frac{1}{2}\left(-\mathrm{Im}(c)\right)\sigma_x^2\right)$$

$$= \cancel{\tfrac{1}{2}}\ \cancel{\tfrac{1}{4}}\ 1/2 \mp \mathrm{Im}(c) = \begin{matrix} 1/3 \\ 2/3 \end{matrix}$$

~~WHICH~~ SO $\mathrm{Im}(c) = \frac{1}{6}$

SO IN THE $|\pm\rangle_y$ BASIS:

$$\rho_y = \begin{pmatrix} 1/3 & \mathrm{Re}(c) + i/6 \\ \mathrm{Re}(c) - i/6 & 2/3 \end{pmatrix}$$

TO BE A PROPER DENSITY MATRIX WE REQUIRE

$Tr(\rho^2) \leq 1$

WHERE $Tr(\rho^2)$

$$= \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2$$

$$+ \underbrace{Re(c)^2}_{\cancel{(1/6)}} + \underbrace{Im(c)^2}_{1/6}$$

$$= \frac{1}{9} + \frac{4}{9} + \frac{1}{36}$$

$$= \frac{4+16+1}{36} = \frac{21}{36}$$

SO $|Re(c)| \leq \sqrt{\frac{21}{36}} = \sqrt{\frac{7}{12}}$

$$\langle S_y \rangle = \cancel{Tr(S_y \rho} \; Tr(\rho S_y)$$

$$= \frac{\hbar}{2} Tr\left[\left(\frac{1}{2}\mathbb{1} - \frac{1}{6}\sigma_y + \cdots\right]\sigma_y\right]$$

(SEE ABOVE ABOUT PAULI MATRIX PROPS)

$$= \frac{\hbar}{2} \cdot \left(\frac{-2}{6}\right) = \frac{-\hbar 2}{12} = \frac{-\hbar}{6}$$

WHICH IS CONSISTENT W/

$$\frac{1}{3}\left(\frac{+\hbar}{2}\right) + \frac{2}{3}\left(\frac{-\hbar}{2}\right) = \frac{-\hbar}{6}$$