302L S20 Lecture 11 - Capacitor energy

Summary:

- The energy E on a capacitor with capacitance C at a voltage V (and thus holding a charge $Q = CV$) is given by

$$\frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV$$

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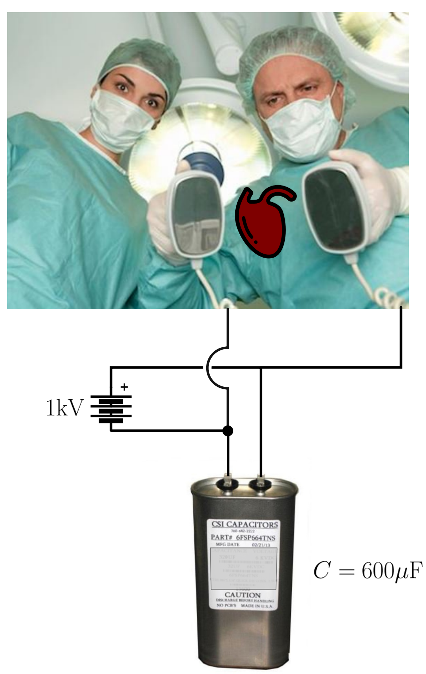
In this lecture we look at how capacitors can store energy. For example, to shock a malfunctioning heart back into action doctors use a defibrillator. A defibrillator is essentially a voltage source attached to a relatively large capacitor:



Conducting wires connect the two ends of the capacitor to a pair of paddles, which are also conductive. Normally the paddles are separated by air, which is insulated, so the capacitor remains charged up at the voltage of the source (roughly 1kV). However, your body, which is a sack of salty water, is conductive, so when the paddles are placed on the left and right sides of a patient's chest, the capacitor becomes "short-circuited", and all the energy stored in the capacitor becomes deposited in the patient's heart. Though it is not fully understood why, this sudden transfer of electrical energy into the heart can restore an improperly beating heart back into normal healthy operation.

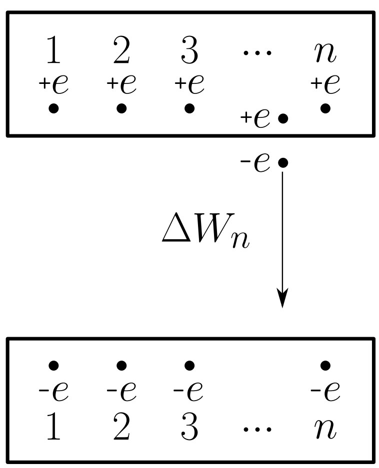
Intuitively we have the impression that in order to get enough energy to defibrillate a heart, we need:

- a lot of charge Q,
- stored at a high potential V.

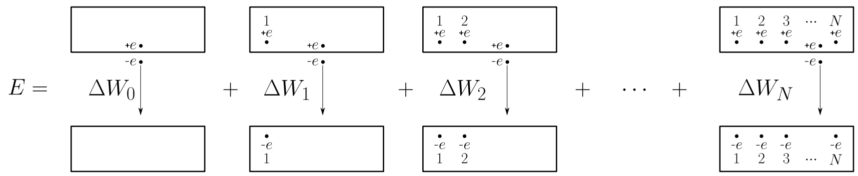
We will see that indeed the energy stored on a capacitor is proportional to the *product* QV of these quantities.

To start with, we make more precise the question of "how much energy is stored on a capacitor?". A more well-formulated question might instead be, "how much *work* W does it take to charge up a capacitor?". We will take this necessary work W to be what we mean by the energy E stored on a capacitor.

The procedure for determining the W is simple. We are going to first figure out the work ΔW_n required takes to transfer a single electron from the positive plate of a capacitor to its negative plate when there are n electrons on the negative plate:



The total work W in transferring N electrons from the positive plate to the negative plate is then simply the sum $\Delta W_1 + \Delta W_2 + \cdots + \Delta W_N$ of the work needed to transfer each electron, one by one:



To calculate ΔW_n, we first note that the work *required*[1] to move a particle from a point A to a point B is equal to the difference $U_B - U_A \equiv \Delta U$ in the particle's final and initial potential energies. The change in the potential energy of a charged particle however is the product $q\,(V_B - V_A)$ particle's charge q and the difference $\Delta V = V_B - V_A$ in the electric potential between the start and end points. Therefore, the work $\Delta W_{(+)\to(-)}$ required to take an electron from the positive (+) plate of a capacitor to the negative (-) plate is:

$$\Delta W_{(+)\to(-)} = (-e)\,(V_- - V_+) = e\,(V_+ - V_-)$$

But if a capacitor with capacitance C has a potential difference $V_+ - V_-$ between its positive and negative plates, then we know the capacitor is holding a charge $Q = C\,(V_+ - V_-)$ on its positive plate. Substituting this in we get

$$\Delta W_{(+)\to(-)} = \frac{eQ}{C}$$

Next we note that if we have n electrons on the negative plate, then the charge $-Q$ on the negative plate is just $-ne$: the number of electrons n times the charge $-e$ of an electron. Therefore, the work ΔW_n required to move an electron from the positive to negative plate, when there are already n electrons on the negative plate, is

$$\Delta W_n = \frac{e(-ne)}{C} = \frac{e^2}{C} n$$

So we find that it takes more work to transfer an electron to the negative plate when there are already a lot of other electrons there. This should not be too surprising since the repulsion away from the negative plate and attraction back towards to the positive plate both get stronger with increasing n.

We're almost done! To determine the energy E on a capacitor with N electrons on the negative plate, we sum up the little works $\Delta W_1 + \Delta W_2 +$

[1]Note this is the opposite of the work *done on* the particle. This is simply energy conservation – the energy given to the particle must have come from somewhere.

$\cdots + \Delta W_N$ required to transfer each electron, i.e.

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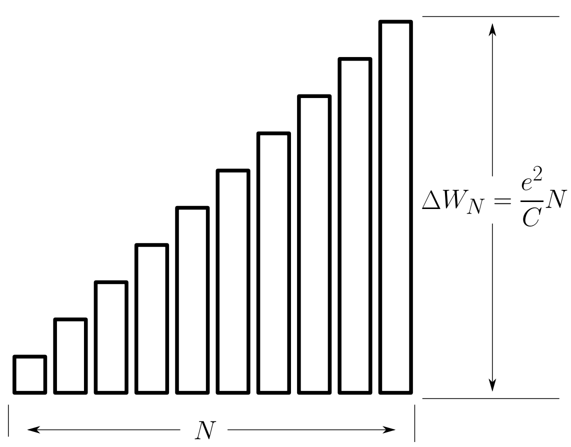
\begin{aligned}
E =& \Delta W_1 + \Delta W_2 + \cdots + \Delta W_N && (1)\\
=& \frac{e^2}{C}1 + \frac{e^2}{C}2 + \cdots \frac{e^2}{C}N && (2)\\
=& \frac{e^2}{C}\left(1 + 2 + \cdots + N\right) && (3)\\
(!) \quad =& \frac{e^2}{C}\left(\frac{N(N+1)}{2}\right) && (4)\\
(*) \quad \approx& \frac{e^2}{C}\left(\frac{N^2}{2}\right) && (5)\\
=& \frac{1}{2}\frac{(Ne)^2}{C} && (6)\\
(\sim) \quad =& \frac{1}{2}\frac{Q^2}{C} && (7)
\end{aligned}
$$

Here is a detailed explanation of the tricky steps:

- (!): In homework 3 you learned that the sum of all the positive whole numbers from 1 to N is $\frac{N(N+1)}{2}$.
- ($*$): Since the charge $-e$ of an electron is so small, any realistic situation will involve so many electrons N that N and $N-1$ are essentially equal.
- ($\sim$): If we have N electrons on the negative plate, then the charge Q on the positive plate is Ne.

To see this another way, we can represent each ΔW_n by a rectangle of width 1 and height W_n and arrange them in increasing order like so:



$$E \approx \frac{1}{2}N\Delta W_N = \frac{1}{2}\frac{e^2N^2}{C}$$

The total work (and thus the capacitor energy) should be the area of all the rectangles, but with the way we've arranged them we can see that our array closely approximates a triangle of base N and height $\Delta W_N = \frac{e^2}{C}N$. From the formula for the area of a triangle ($\frac{1}{2}$base $\times$ height) we reproduce the formula for the capacitor energy.

So our work here is essentially done. If we want to know the energy E of a capacitor with a capacitance C holding a charge $\pm Q$, we simply plug the values into equation (1).

However, equation (1) is often not very convenient. Typically we are given the voltage V across a capacitor, and do not initially know the charge Q. This is because sources of constant voltage (batteries, wall outlets, cell phone chargers, etc.) are quite common, but source of constant charge are pretty rare.

It would be nice then to have an expression for the capacitor energy in terms of its voltage V (and capacitance C). Well we know that $C = \frac{Q}{V}$, or in other words $Q = CV$, so plugging this into equation (1) we get:

$$E = \frac{1}{2}\frac{(CV)^2}{C} = \frac{1}{2}CV^2 \tag{8}$$

At the beginning of the lecture I claimed that the capacitor energy was proportional to the product of the charge Q and the potential V. To see this we make a single substitution $CV \to Q$ into the previous equation:

$$E = \frac{1}{2}(CV)\,V = \frac{1}{2}QV \tag{9}$$

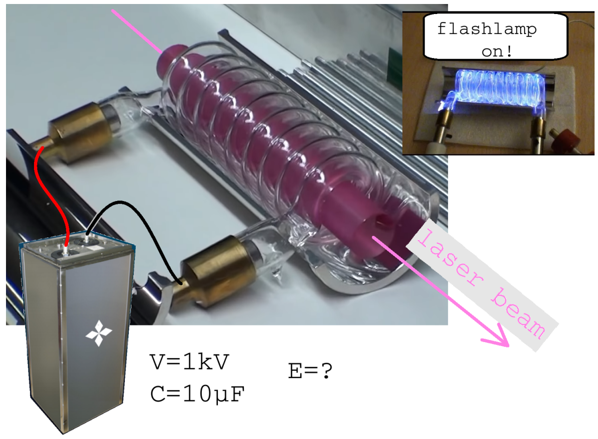
Any of equations (1), (8), or (9) can be used to calculate the capacitor energy[2]. Use whichever one is most convenient for the problem at hand.

Example: Flashlamp + Laser

One scenario where we need to store a lot of electrical energy is in the generation of pulsed lasers. In my lab on the bottom floor of the physics building we have a pulsed laser that produces about 1 Joule of laser light that is only 5ns $= 5 \cdot 10^{-9} s$ in duration. To initiate the generation of a laser pulse, a 100μF capacitor which has been charged up to 1kV is discharged through a "flash lamp" which is wrapped around a crystal[3]. The crystal absorbs energy from the flash lamp and reradiates it as laser light. See the picture below for an illustration:

[2] The relation $E = \frac{1}{2}QV$ might look odd since we know that for a particle of charge q in a potential V we have $U = qV$. Why then do we have a factor of $\frac{1}{2}$ here? The answer is that in the relation $U = qV$ we assume that the electric potential was constant as our particle moved between two points. In calculating the energy of a capacitor, however, we needed to take into account the fact that the potential was *increasing* every time we move an electron between the positive and negative plates. We do not expect then for the two expressions to agree, since they correspond to different physical situations.

[3] The flash lamp is basically a neon (actually xenon) sign that is only on for a few microseconds. Special capacitors are needed in order to get a discharge this rapid.



We could ask about the efficiency of this arrangement: what percentage of the energy that we put into the flash lamps comes out of the crystal as laser light?

The energy E_C stored in the capacitor is given by

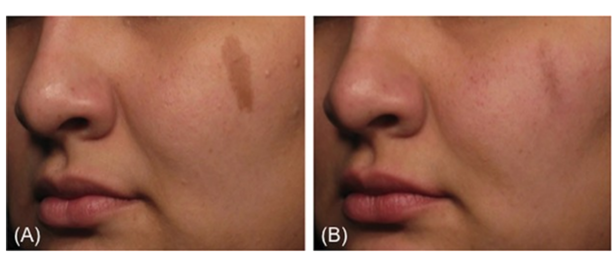
$$\begin{aligned} E_C =& \frac{1}{2} C V^2 \\ =& \frac{1}{2} (100\mu\text{F}) (1\text{kV})^2 \\ =& \frac{1}{2} 100 \times 10^{-6}\text{F} \times \left(10^3 \text{kV}\right)^2 \\ =& \frac{1}{2} 100 \times 10^{-6}\text{F} \times 10^6 \text{V}^2 \\ =& \frac{100}{2}\text{J} = 50\text{J} \end{aligned}$$

The efficiency is then the ratio of the laser pulse energy $E_L = 1\text{J}$ and the capacitor energy:

$$\text{efficiency} = \frac{E_L}{E_C} = \frac{1\text{J}}{50\text{J}} = 2\%$$

Not a terrible conversion efficiency (definitely don't want to get hit in the eye with 1J of laser energy).

These pulsed laser incidentally find application in the field of dermatology, where they are used to remove wrinkles and tatoos, and also treat pigmentary conditions. See this example[4] of the laser treatment of a "Cafe au lait macule", i.e. a birthmark:



[4] Levin et. al, Lasers in Surgery and Medicine 48:181–187 (2016)

Figure (A) shows the birthmark before laser treatment, and figure (B) shows the improvement after a single laser treatment.

Instapoll Question: A defibrillator is made from a 1kV battery and a 600μF capacitor. Approximately how much energy is needed to restore a fibrillating heart to normal operation?

Answer:

$$\frac{1}{2}CV^2 = \frac{1}{2} \times 600 \times 10^{-6}\mathrm{F} \times 10^6\mathrm{V}^2 = \frac{600}{2}\mathrm{J} = 300\mathrm{J}$$

For comparison, that is roughly twice the energy of the fastest baseball every thrown.

Instapoll Question: A capacitor that is connected to some voltage source contains an energy E. The capacitor is then removed from the voltage source and then wired in parallel with an identical capacitor. The parallel combination is then connected to the same voltage source. What is the total energy contained in the two capacitors?

Answer:

Here V is fixed but the effective capacitance of the parallel combination is double the capacitance of our original capacitor (capacitances in parallel add together). Therefore:

$$E' = \frac{1}{2}(2C)V^2 = 2\left(\frac{1}{2}CV^2\right) = 2E$$

This makes sense since we do not expect to get any more energy wiring two capacitors to the same battery than by wiring each capacitor to its own separate battery (of equal potential). You don't get something for nothing :)