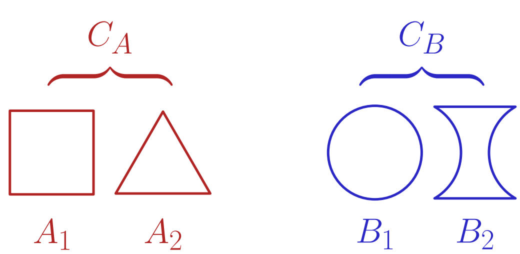
302L S20 Lecture 10 - Capacitors in parallel and in series

1 Summary

- Capacitors in parallel:
  - By combining two capacitors with capacitances C_A and C_B in parallel we obtain a new capacitor with capacitance $C_A + C_B$
  - The charges Q_A and Q_B on capacitors C_A and C_B joined in parallel are proportional to their capacitances; i.e. $Q_A/Q_B = C_A/C_B$.
- Capacitors in series:
  - By combining two capacitors with capacitances C_A and C_B in series we obtain a new capacitor with capacitance $\left(\frac{1}{C_A} + \frac{1}{C_B}\right)^{-1}$
  - The charges Q_A and Q_B on capacitors C_A and C_B joined in series are the same and equal to the total charge Q across the series combination.

2 Capacitors in parallel

Consider two capacitors, shown below



Capacitor A is composed of two conductors A_1 and A_2 and has a capacitance C_A, and likewise for capacitor B. Recall that when we say a capacitor A, i.e. a conductor pair, has some capacitance C_A, we mean that:

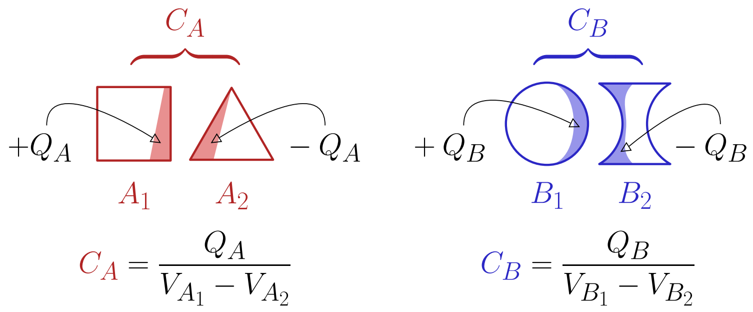
- if we were to place a charge $+Q_A$ on one of the conductors (A_1) and an opposite charge $-Q_A$ on the other (A_2),
- the charge will rearrange itself until

- the electric potential at conductor A_1 is a constant V_{A_1} and
- the electric potential at conductor A_2 is a constant V_{A_2}, and

- the potential difference $V_{A_1} - V_{A_2}$ between the two conductors is given by

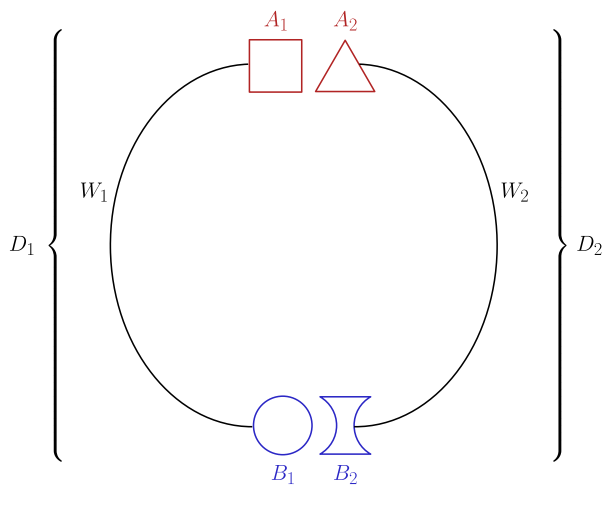
$$V_{A_1} - V_{A_2} = \frac{Q_A}{C_A}$$

The graphic below illustrates this:



The light shaded regions are meant to illustrate roughly how the surface charge distributes itself to achieve a constant potential at each conductor[1]. Keep in mind that in the above we are assuming the two conductors are in isolation – i.e. that there are no other charges or conductors around.

Now suppose we join together conductors A_1 and B_1 with a long fine conducting wire W_1, and likewise for A_2 and B_2:



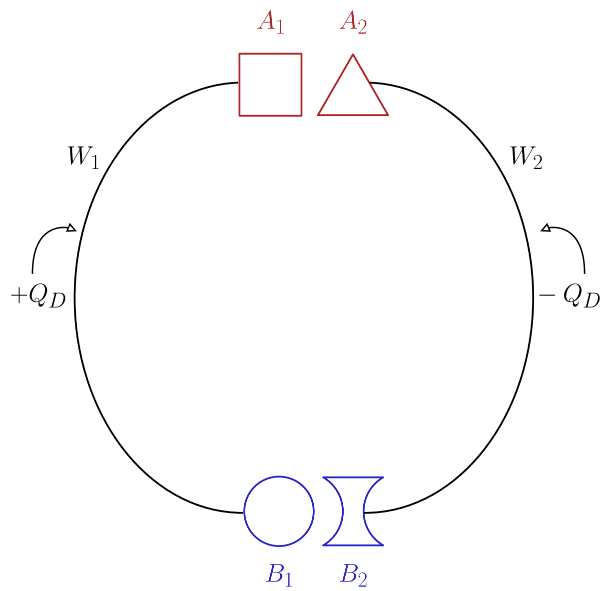
What we end up with is a pair of conductors D_1 and D_2 where:

- D_1: A_1, B_1, and W_1
- D_2: A_2, B_2, and W_2

[1]Keep in mind though that in reality all the charge on a conductor lies on its *surface* and never *inside* the conductor.

So we find we have taken a pair of capacitors and made another capacitor which we can call D. What is the capacitance C_D of this our new capacitor?

To determine this, we place some charge $+Q_D$ on D_1 and $-Q_D$ on D_2:

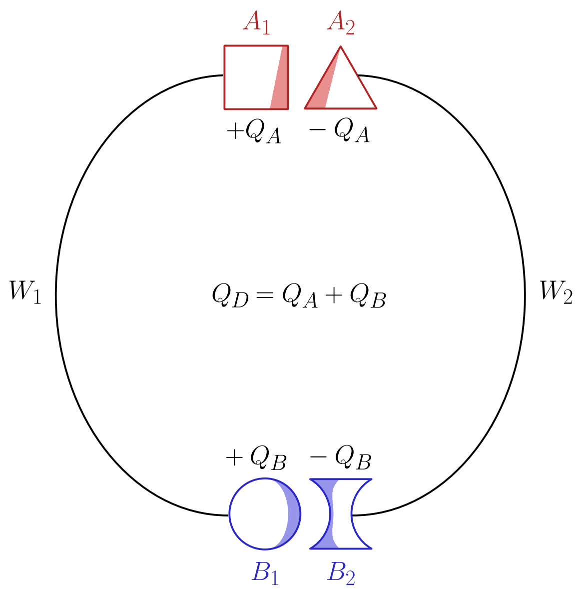


and then figure out how this charge distributes itself in order to keep each conductor at a constant potential.

Once the charge equilibriates, we do not expect the wires (W_1 and W_2) to contain much charge, for two reasons:

- The wires are fine and so putting significant charge on them would result in large mutual repulsion.
- The two wires are far apart from one another. Any positive charge on W_1 and negative charge on W_2 could reduce their separation by moving to one of the capacitors A or B.

So to a good approximation we can say the charge $\pm Q_D$ splits up into $\pm Q_A$ and $\pm Q_B$ (i.e. $Q_D = Q_A + Q_B$) and collects on the two capacitors A and B:



How does the charge divide up? Well in equilibrium conductors A_1 and B_1 are the same potential, i.e

$$V_{A_1} = V_{B_1}$$

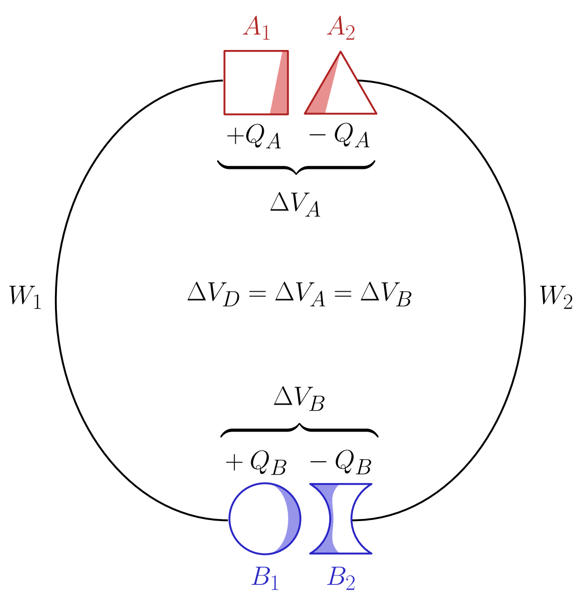
because they are joined together by the conducting wire W_1, and likewise for A_2 and B_2, i.e.

$$V_{A_2} = V_{B_2}$$

Therefore the *difference* $V_{A_1} - V_{A_2} \equiv \Delta V_A$ must equal the difference $V_{B_1} - V_{B_2} \equiv \Delta V_B$, which are both equal to the potential difference ΔV_D across our new capacitor D, i.e.

$$\Delta V_A = \Delta V_B \equiv \Delta V_D \tag{1}$$

We can illustrate this like so:



Since we have separated the capacitors by some large distance, we can also assume that, to a good approximation, the capacitance C_A of capacitor A is unaffected by the presence of capacitor B, so that:

$$Q_A = C_A \Delta V_A = C_A \Delta V_D \tag{2}$$

and likewise

$$Q_B = C_B \Delta V_B = C_A \Delta V_D \tag{3}$$

2.1 Aside – adjustable capacitor demo

Pausing for a moment in our calculation of the capacitance C_D of our new capacitor, we can rearrange the above equations (2) and (3) to obtain

$$\frac{Q_A}{Q_B} = \frac{C_A}{C_B}$$

we find then the very sensible result that when we connect two capacitors in parallel and charge them, the charge will prefer to flow onto the capacitor with the larger capacitance[2].

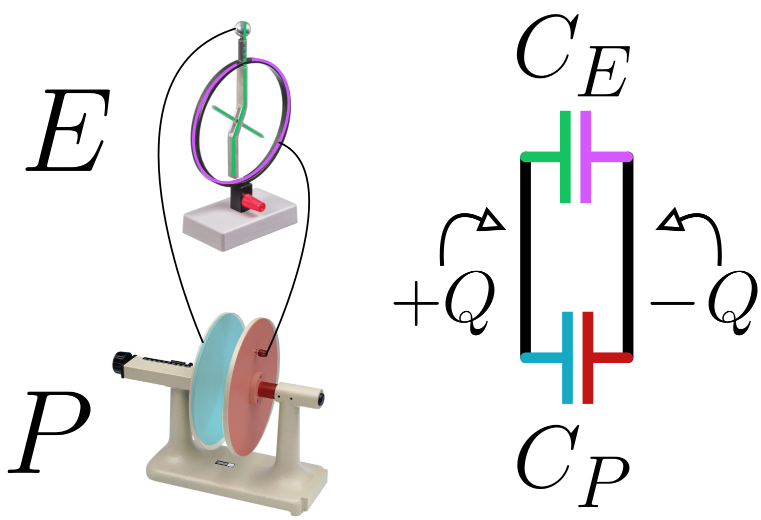


Figure 1: Illustration of adjustable capacitor demo.

To demonstrate this principle, we have connected in parallel two two capacitors. See figure 1 for reference. The two capacitors are:

1. A parallel plate capacitor, which we will refer to as P, with an adjustable separation.
2. An electroscope, which you might remember from the first couple lectures. The electroscope consists of two conductors, held together by insulating material. The two conductors are:
   - a metal ring (purple), which surrounds

[2]See the end of section 4 in the previous lectures notes for a discussion of this connection between capacitance and "charge affinity" in the context of the parallel plate capacitor.

- a vertical bar with a slot for a thin needle to swivel about (green).

When the electroscope becomes charged, the needle tilts away from its normal vertical orientation so that the charges it carries can maximize their separation from the like charges held by vertical bar. Therefore the extent of the needle's tilt gives a measure of the charge on the electroscope.

Let's begin with a large spacing between the plates of our parallel plate capacitor. According to the formula for the capacitance of a parallel plate capacitor derived in the previous lecture:

$$C = \frac{\epsilon_o A}{d}$$

a large d implies a smaller capacitance than a larger d.

Using the Wimhurst machine we can charge up our capacitor pair with a charge $\pm Q$ large enough to observe deflection of the electroscope needle. We then disconnect the Wimhurst machine from our capacitor pair, so that the pair is left with some fixed charge. Since charge is conserved, this charge is stuck on the capacitor pair forever (or until the humid Austin air provides a path for the opposite charges $\pm Q$ to recombine – whichever happens first).

Instapoll question:

If we reduce the separation between the plates on the parallel plate capacitor, what will happen to the electroscope needle?

1. Deflects vertically
2. Deflects horizontally
3. Nothing

Answer:

As we decrease the plate separation, the capacitance C_P increases. Since charge prefers to large capactance, the charge will flow from electroscope, whose capacitance C_E is fixed throughout, onto the parallel plate capacitor. The reduction in charge on the electroscope can be observed by a deflection of the electroscope towards its uncharged state, i.e. in the vertical direction.

2.2 Parallel capacitance derivation resumed

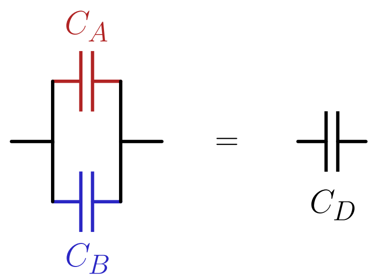
To finish our calculation for the capacitance C_D, we add together the above expressions for Q_A and Q_B (equations (2) and (3)), obtaining

$$Q_D = Q_A + Q_B = C_A \Delta V_D + C_B \Delta V_D = (C_A + C_B)\, \Delta V_D$$

dividing both sides by ΔV_D we find

$$\frac{Q_D}{\Delta V_D} = C_A + C_B$$

We find then that by connecting two capacitors in parallel we obtain a new capacitor with a capacitance equal to the *sum* of the capacitances of the original capacitors. This is illustrated schematically by the following diagram:



$$C_D = C_A + C_B$$

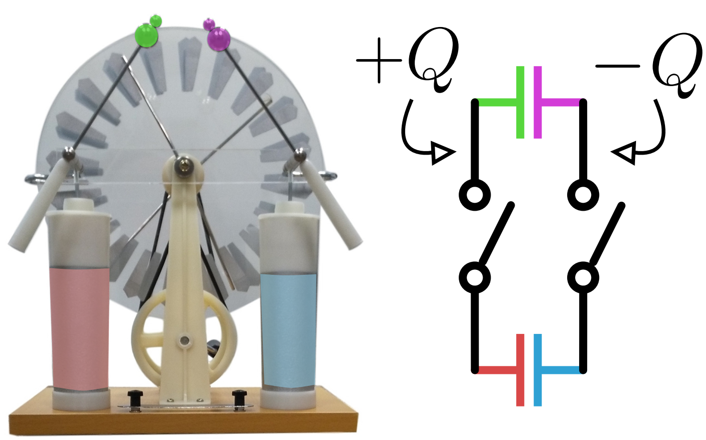


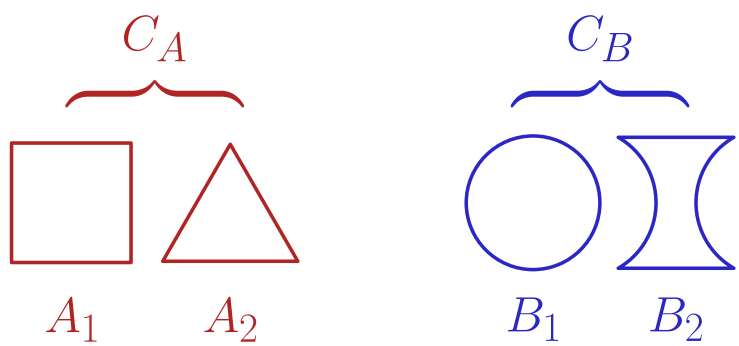
Figure 2: Wimhurst machine with Leiden jars.

By observing the sparks generated from the Wimhurst machine we see a simple example of parallel capacitors in action. A diagram of the set up is shown in figure 2. The Wimhurst machine transfers equal and opposite charge onto the left and right metal spheres, shown in green and purple on the diagram. Charge builds up on the spheres until the potential difference reaches some threshold voltage beyond which the air between them becomes conductive and the spheres discharge with a visible and audible spark. The brightness and loudness of the spark is determined by the size of the discharge. Because the capacitance of the pair of spheres is quite small, it does not take very much charge to reach this threshold voltage and therefore the spark is dim and quiet.

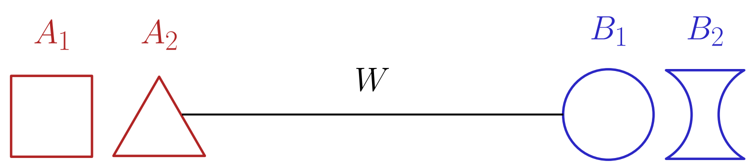
By connecting two "Leiden jars", shown in red and blue, in parallel with the spheres we add onto the small capacitance of the spheres the much larger capacitance of the Leiden jar pair. The result is that the sparks are now much brighter and louder. Also note how the sparks are spaced out by longer intervals, indicating that more time is required to charge this parallel capacitance up to the threshold voltage.

3 Capacitors in series

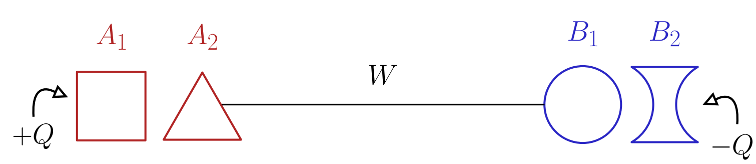
Now let's take our capacitors A and B again:



and this time only connect A_2 and B_1 with a long fine wire W, leaving A_1 and B_2 disconnected:

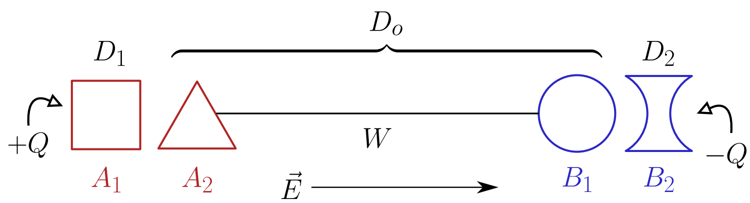


Connecting the capacitors in this way is known as connecting them *in series*. However, we now have three conductors, so, as it stands, it is not yet clear in what sense we have combined capacitors A and B to form another capacitor D with some capacitance C_D. What we will find is that, by placing a charge $+Q$ and opposite charge $-Q$ on the unconnected conductors A_1 and B_2:

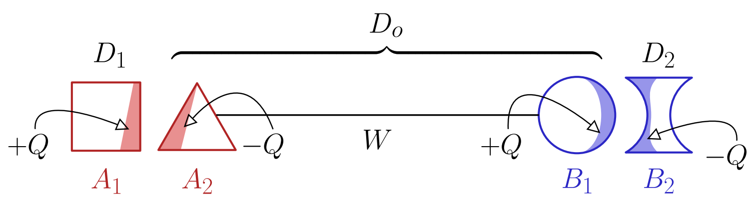


we generate a voltage $V_{A_1} - V_{B_2}$ across the two conductors which is proportional to Q, so that the ratio $\frac{Q}{V_{A_1}-V_{B_2}} \equiv C_D$ is *constant*, i.e. independent of Q. Therefore, if we rename $A_1 \to D_1$ and $B_2 \to D_2$, we obtain a capacitor D with a capacitance C_D which we call the *series capacitance* of capacitors A and B

So it remains to show you what I claimed: that this ratio $\frac{Q}{V_{A_1}-V_{B_2}}$ which we are calling C_D is indeed constant. The key is to see that when we first place the charges $\pm Q$ on A_1 and B_2, we generate an electric field $\vec{E}$ along the trio D_o of connected conductors A_2, W, and B_1:



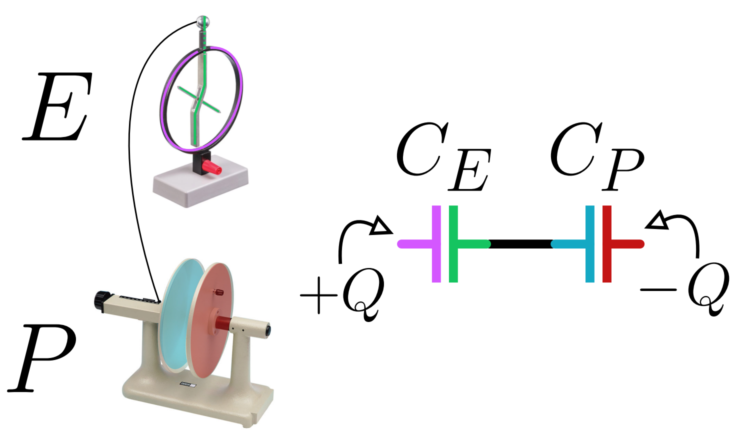
This arrangement is not stable ($\vec{E}$ must be zero everywhere inside a conductor at equilibrium). Positive charge carriers in the conductor move with the electric field and negative charge carriers move oppositely until the electric potential is constant everywhere along D_o. This charge transfer occurs until we get $-Q$ on A_2 and $+Q$ on B_1:



Let's pause again to contrast the parallel and series combinations. In the parallel case we argued that the $\pm Q$ placed on the parallel combination distributed itself among the two capacitors A and B so that $\frac{Q_A}{Q_B} = \frac{C_A}{C_B}$.

In the series case we have arrived at a very different conclusion. Here we find that putting $\pm Q$ on the unconnected conductors D_1 and D_2 resulted in a spontaneous *polarization* (charge separation) of the connected conductors D_o so that the charges Q_A and Q_B on capacitors are both equal to the full charge Q that we placed on our series capacitor. This conclusion holds for any value of the capacitances C_A and C_B.

With this in mind, let's repeat the experiment with the adjustable parallel plate capacitor and electroscope, this time connecting the two in series:



Instapoll question:

What will happen to the electroscope needle this time when we reduce the plate separation on the parallel plate capacitor?

Answer:

Since the charge on the unconnected conductors (purple and red) stays fixed at $\pm Q$ (it has nowhere else to go!), we find that the charge across the electroscope must also stay fixed at $\pm Q$, no matter what changes we make to the parallel plate capacitor. Since the needle deflection is a measure of the charge on the electroscope, we conclude that we expect *no* change in the needle deflection.

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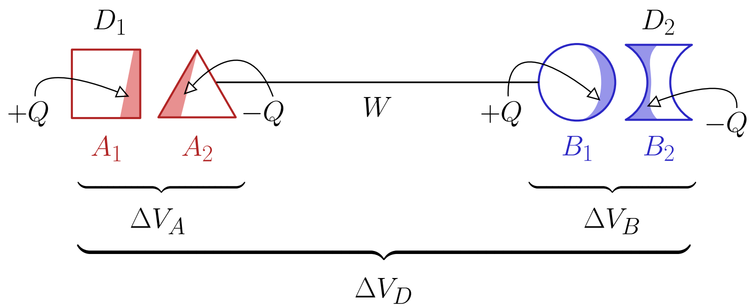
Continuing our derivation for the series capacitance, we note that, as things now stand, all the conductors are at equilibrium. Therefore, the potential at A_2 is the same as the potential at B_1, since they are part of the same conductor. We can then write the voltage difference $V_{D_1} - V_{D_2} \equiv \Delta V_D$ across our series capacitor in the following way:

$$\begin{aligned}
\Delta V_D =& V_{D_1} - V_{D_2} \\
=& V_{A_1} - V_{B_2} \\
=& V_{A_1} - (V_{A_2} - V_{B_1}) - V_{B_2} \\
=& (V_{A_1} - V_{A_2}) + (V_{B_1} - V_{B_2}) \\
\equiv& \Delta V_A + \Delta V_B
\end{aligned}$$

In the figure below we illustrate this graphically:



But if our capacitors A and B are far enough away from each other that they do not influence each others' capacitance, then we can also say

$$\Delta V_A = \frac{Q}{C_A}$$

and likewise

$$\Delta V_B = \frac{Q}{C_B}$$

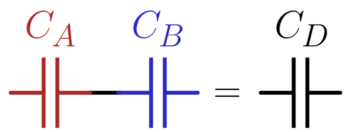
so that

$$\Delta V_D = \Delta V_A + \Delta V_B = Q\left(\frac{1}{C_A} + \frac{1}{C_B}\right)$$

After some rearranging we find

$$C_D = \frac{Q}{\Delta V_D} = \left(\frac{1}{C_A} + \frac{1}{C_B}\right)^{-1}$$

So that our ratio C_D is, as claimed, independent of Q. In the end we find that taking two capacitors and connecting them in series we obtain a new capacitor with a capacitance given by the reciprocal of the sum of their reciprocals. That is quite a mouthful so lets state the answer schematically as we did for the parallel combination:



$$C_D = \left(\frac{1}{C_A} + \frac{1}{C_B}\right)^{-1}$$

The series capacitance definintely has a trickier formula than the parallel capacitance, so let's do some examples to get a feel for things.

Example:

What is the series capacitance of two capacitors with the same capacitance C?

Answer: Plugging in $C_A = C = C_B$ into the expression for series capacitance we get

$$\left(\frac{1}{C} + \frac{1}{C}\right)^{-1} = \left(\frac{2}{C}\right)^{-1} = \frac{C}{2}$$

so that putting two identical capacitors in series results in a series capacitance with *half* the capacitance of the original capacitors.

Example:

What is the series capacitance of two parallel plate capacitors with equal plate area A and plate separations d_1 and d_2?

Answer:

From the formula $C = \frac{\epsilon_o A}{d}$ for the parallel plate capacitance and the formula for the series capacitance we get:

$$\left(\frac{d_1}{\epsilon_o A} + \frac{d_2}{\epsilon_o A}\right)^{-1} = \left(\frac{d_1 + d_2}{\epsilon_o A}\right)^{-1} = \frac{\epsilon_o A}{d_1 + d_2}$$

so that we effectively end up with another parallel plate capacitor with a separation equal to the sum of the individual separations.

Instapoll question:

What is the approximate series capacitance of two capacitors A and B with capacitances C_A and C_B, where $C_A \gg C_B$?

Answer:

If $C_A \gg C_B$, then $\frac{1}{C_A} \ll \frac{1}{C_B}$. Therefore $\frac{1}{C_A} + \frac{1}{C_B} \approx \frac{1}{C_B}$. If we take our formula for the series capacitance:

$$C_D = \left(\frac{1}{C_A} + \frac{1}{C_B}\right)^{-1}$$

and plug in our approximation we get

$$C_D \approx \left(\frac{1}{C_B}\right)^{-1} = C_B$$

This means that if we can take a small capacitance and put in series with it a large capacitance and end up with essentially what we started with.