302L S20 Lecture 9 - Capacitors

1 Summary

- In equilibrium, the electric potential is the same everywhere on a conductor.
- A capacitor is a pair of conductors.
- The capacitance C of a conductor is the ratio Q/V where V is the potential difference between the conductors when there is a charge Q on one conductor and an opposite charge $-Q$ on the other.
- In SI units capacitance is given in Coulombs per Volt, i.e. C/V. A capacitance of 1 C/V is known as a Farad, abbreviated F.
- The capacitance of a parallel plate capacitor is $\frac{\epsilon_o A}{d}$, where A is the area of the plates and d is the distance between them.
- Capacitors are used to store charge.

2 Electric potential at conductors

From lecture 6 we learned about the electric field inside of and at the surface of conductors. We found that, in equilibrium, the electric field and the charge density inside a conductor is always zero. Any charge resides entirely on the surface and any electric field at the surface points in the direction perpendicular to the surface.

What can we say then about the electric *potential* $V(\vec{x})$ at a conductor? The rule turns out to be very simple: in equilibrium, the electric potential is the *same* everywhere on the conductor – even the surface. To see this let's refer to diagram 1.

Equation (2) from lecture 8 says that the potential difference $V_2 - V_1$ between two points $\vec{x}_1$, $\vec{x}_2$ in a constant electric field $\vec{E}_o$ is given by:

$$V_2 - V_1 = -\vec{E}_o \cdot (\vec{x}_2 - \vec{x}_1)$$

Well, an electric field of *zero*[1] is certainly constant, so applying the above equation we find that the electric potential between *any* two points (e.g. $A \to B$, $A \to C$ in the figure) on or in the conductor is zero!

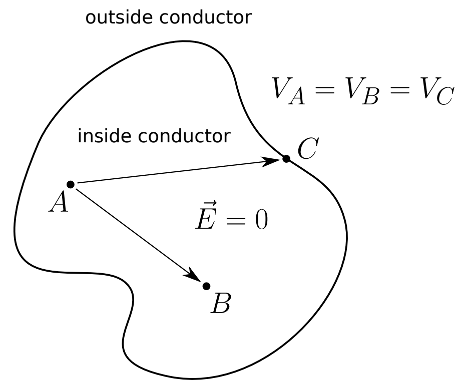

Figure 1: The electric potential is the same for any point on or in a conductor.

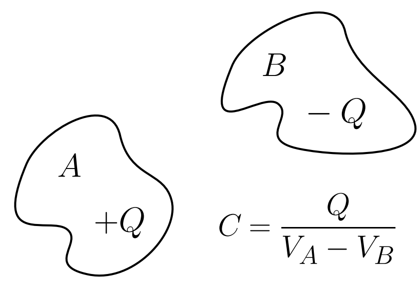

Figure 2: A capacitor is formed by a pair of conductors.

3 Capacitors

Let's take two conductors, A and B, and place some charge $+Q$ on A and some opposite charge $-Q$ on B (see figure 2). The combination of:

- the mutual repulsion between the charge carriers within each conductor, and
- the attraction of the charge on A to the opposite charge on B

will result in a distribution of the charge on the surface of each conductor so that the electric potential is a constant value V_A everywhere on A and a constant value V_B on B, as is required of conductors.

Now if we add another $+Q$ of charge to A and another $-Q$ to B, then this added charge will distribute itself in exactly the same way as the original charge. By the superposition principle then we find that doubling the charge on each conductor simply doubles potential difference $V_A - V_B$ between the two conductors.

In other words, the *ratio* $\frac{Q}{V_A - V_B}$ is fixed for a given pair of conductors. This ratio is termed the *capacitance* C of the conductor pair, i.e.

$$C = \frac{Q}{V_A - V_B}$$

[1]i.e. $\vec{E}_o = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$

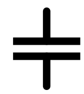

Figure 3: Symbol for a capacitor

This is more commonly expressed using the shorthand

$$C = \frac{Q}{V} \tag{1}$$

where V is understood to be the increase in electric potential going from the conductor with charge $-Q$ to the conductor with charge $+Q$.

The conductor pair itself is termed a *capacitor*. In electrical circuit diagrams (i.e. "schematics"), capacitors are represented by the symbol shown in figure 3. Each black "T" is meant to represent one of the conductors in the conductor pair.

From the above equation we see that capacitance is a charge divided by a voltage. The SI units for capacitance is therefore in Coulombs per Volt, or C/V. This quantity of 1 C/V is known as a *Farad* (abbreviated F), named after the physicist Michael Faraday.

Dessert platter of capacitances:

- human body + the floor:

  150pF $= 150 \cdot 10^{-12}$F

- inside + outside human cell membrane:

  1nF $= 1 \cdot 10^{-9}$F

- subwoofer power supply capacitor:

  3 F

- two electrodes on a computer microprocessor transistor:

  3fF $= 3 \cdot 10^{-15}$ F!

4 Parallel plate capacitor

To determine the capacitance of a conductor pair A and B, it appears we need to:

1. put some charge $+Q$ on A and some opposite charge $-Q$ on B, then
2. figure out how the charge needs to distribute itself so that the electric potential is a constant V_A on A and a constant V_B on B, then
3. take the ratio $\frac{Q}{V_A - V_B}$ to get C

This is in general a hard problem, but we can figure it the solution for some simple cases.

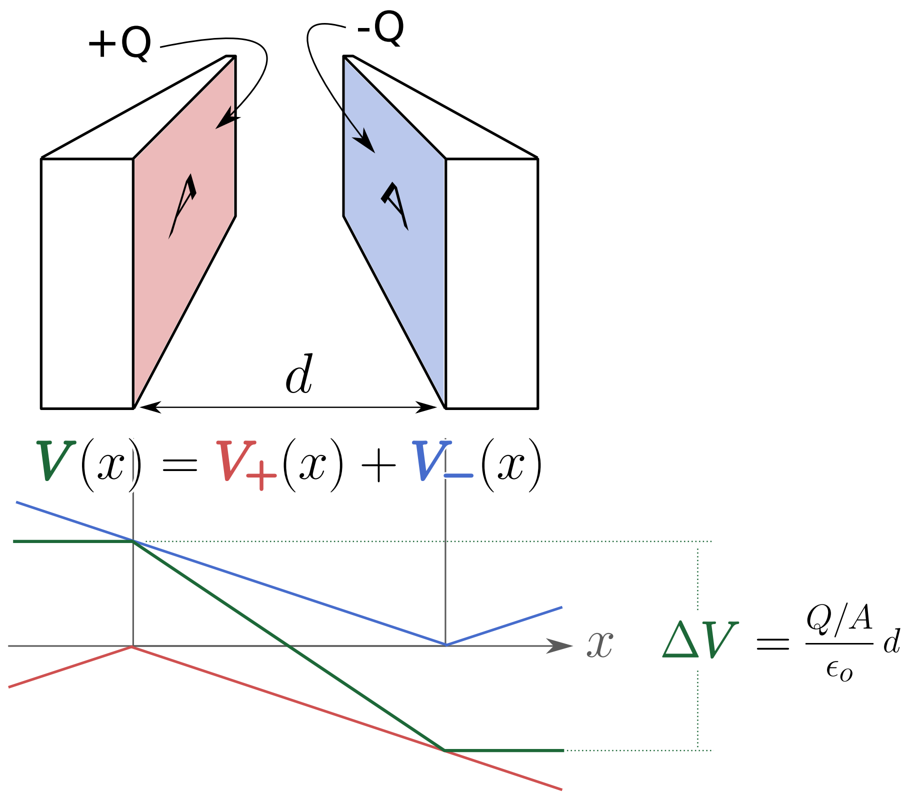

Figure 4: Parallel plate capacitor

We will work out the example of the *parallel plate capacitor*, which consists of two identical plates of area A separated by a distance d. See figure 4 for an illustration.

In the figure we have a perspective view looking between the two plates. The left and right plates have a charge $+Q$ and $-Q$ respectively. If the separation d between the plates is small compared to the dimension of the plates (e.g. their radius if they're circular or their side length if they're square), then I claim that the charge $+Q$ on left plate will distribute itself evenly over the face that faces the right plate, and likewise for the charge $-Q$ on the right plate. This assumption is reasonable because

- the postive charge and the left plate is drawn towards the negative charge on the right (and vice versa), and
- the like charges on a given plate will repel one another, spreading out to maximize their separation.

For points near the centers of the plates the electric potential can be approximated by the potential generated by two *infinite* sheets of surface charge density $\sigma_\pm = \pm\frac{Q}{A}$ separated by a distance d. The electric potential $V_+(x)$ generated by the left plate at these points is, by formula (5) in lecture 8:

$$V_+(x) = -\frac{\sigma_+}{2\epsilon_o}x = -\frac{Q/A}{2\epsilon_o}x$$

where x is your distance from the plate. This potential is plotted in red on the graph below the plates in the figure. The electic potential $V_{-}(x)$ (plotted in blue) due to the right plate looks similar to $V+(x)$ except it is shifted to the right by a distance d and flipped upside down since the plate is oppositely charged.

By the superposition principle, the electric potential generated by both plates is the sum of the electric potentials generated by each plate separately. This sum is shown as the solid green line in the plot. We find that:

- *Inside* the plates, the slopes of the left and right potentials *cancel*, resulting in a constant potential. Thus we find we have distributed our charge on the plates so that we have satisfied the requirement given in step 2 of the procedure outlined in the beginning of the section.

- *In between* the two plates, the slopes for the left and right plates are equal so that the slope of their sum is *double* the slope of either.

- The electric potential *difference* $V(0) - V(d) \equiv \Delta V$ between the two plates is thus given by the slope $\frac{Q/A}{\epsilon_o}$ times the distance d between the plates, i.e.
$$\Delta V = \frac{Q/A}{\epsilon_o} d$$

We find then, after some algebra, that the capactance C of a parallel plate capacitor is given by the simple formula

$$C = \frac{Q}{\Delta V} = \frac{\epsilon_o A}{d} \tag{2}$$

Instapoll question:

A parallel plate capacitor with circular plates has a capacitance C. If we double the radius of the plates, what is the new capacitance?

5 Water flow analogy

So far we have said what a capacitor is, defined its capacitance, and solved for the capacitance of a parallel plate capacitor. But what are capacitors good for? When do we use them?

Capacitors are used mainly for charge *storage*. An analogy at this point may be helpful. Refer to figure 5.

Suppose you are trying to clean an especially dirty pot that your roommate has left in the sink for a couple weeks. To blast away the crud you need a high pressure stream of water from the faucet. The source of water feeding your faucet comes from the town's water tower. The town employs someone to keep the tower water filled up to some minimum height or, equivalently, to some minimum pressure. The water pressure *at the tower* is thus plenty enough to clean the pot, but, by the time the water has travelled down the long narrow pipe connecting the tower to your house, the pressure is significantly reduced so that all you get is a piddling drip.

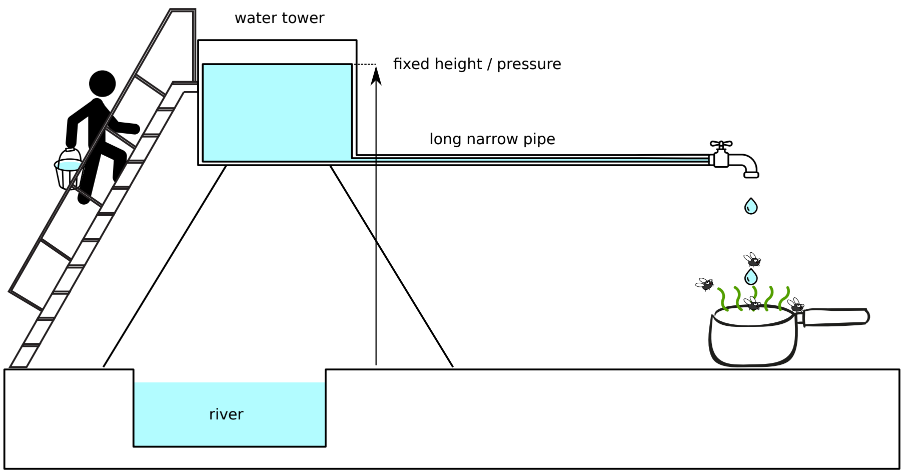

Figure 5: Water pressure is reduced after traveling a long narrow pipe.

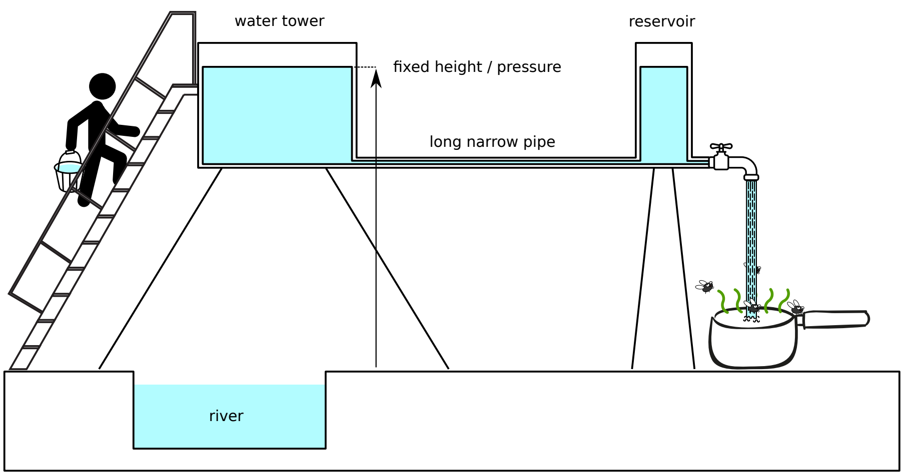

Figure 6: By adding a reservoir you can store large quantities of water at the full water tower pressure.

To solve this problem we add a water reservoir near the faucet. When the faucet is off, the water tower begins filling up the reservoir. There is still the long narrow pipe connecting the two so the reservoir will fill rather slowly, but if we wait long enough the reservoir will fill up to the height of the water tower so that their pressures are equal. Now when we go to turn the faucet on, there is only a short short length of pipe connecting the reservoir to the faucet so we get the full water tower pressure at the faucet and therefore a strong flow of water to clean our pot with.

To connect this analogy to the concepts of electric potential (i.e. voltage) and capacitance, we make the following substitutions:

- The reservoir serves the role of the capacitor. Instead of water, a capacitor stores charge. If we want to store a lot of water, we construct a large reservoir. Similarly, if we want to store a lot of charge, we use a capacitor with a large capacitance.

- Likewise, the water tower serves the role of a *battery*. Instead of being a source of constant pressure, a battery is a source of constant *voltage*. In the same way the water tower employee does work to keep the water tower always at the same pressure, a battery uses chemical energy to always keep constant the electric potential between its positive and negative terminals.

- Though we are jumping the gun a bit, the long narrow pipe is analagous to a large electrical *resistance*. In our example the long narrow pipe prevented a strong flow of water to the place we needed it. Similarly, a large electrical resistance prevents a strong flow of charge from the battery to the electric device which needs it (e.g. light bulb, motor, subwoofer).

- The faucet serves the role of a switch, which can be used to divert the flow of charge. When the faucet is off, the reservoir charges up, and when the faucet is on the reservoir empties out since the tower can not supply water fast enough to replace the water rushing out of the faucet. Likewise, when the switch is open, the battery builds up charge on the capacitor, and when the switch is closed, the charge is drained off of the capacitor.

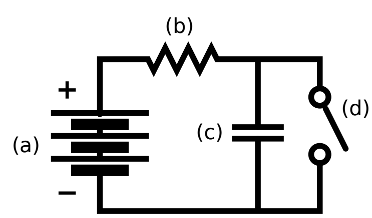

Figure 7: Electrical circuit analog of figure 6. **(a)** battery, **(b)** resistor, **(c)** capacitor, **(d)** switch.

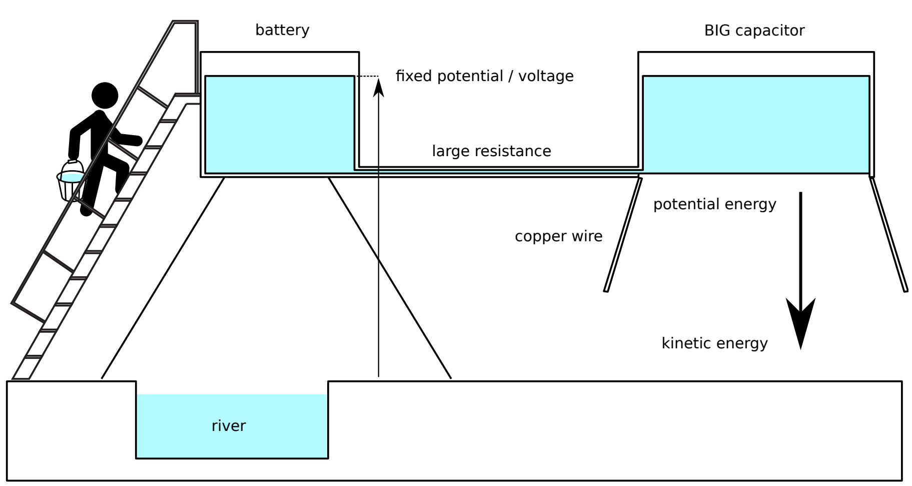

Figure 8: A trap door opens, instantly releasing all the water in the reservoir.

The electrical analog of figure 6 is shown in figure 7. We have a realization of this electrical "circuit" for our demo today. A copper wire is acting as our switch. It begins connected to only one terminal of the capacitor. The switch is thus open, and the capacitory charges up to the full voltage of the supply. Once the capacitor is charged we disconnect it from the power supply and then connect the dangling end of the copper wire to the other terminal of the capacitor.

A more apt version of the water analogy is shown in figure 8. We have a very large reservoir filled up to a high pressure. Then all of the sudden we release a trap down under the reservoir, and in nearly in an instant all the gravitational potential energy is converted into kinetic energy. What might we expect to happen with such a sudden and extreme transfer of matter?