302L S20 Lecture 8 - The Electric Potential

1 Summary

- The electric potential $V(\vec{x})$, like the electric field $\vec{E}(\vec{x})$, is a way of describing the influence of a charge distribution on other charged particles.
- The potential energy $U(\vec{x})$ of a particle with charge q interacting with a charge distribution generating an electric potential $V(\vec{x})$ is given by $U(\vec{x}) = qV(\vec{x})$
- Electric potential has units of volts (V = 1J/C) and is alternatively referred to as a *voltage*.
- The electric potential difference $V_2 - V_1$ between two points $\vec{x}_1$ and $\vec{x}_2$ in a region of constant electric field $\vec{E}_o$ is given by

$$V_2 - V_1 = -\vec{E}_o \cdot (\vec{x}_2 - \vec{x}_1) = -E_o d \cos\theta$$

  where

  - $E_o = |\vec{E}_o|$ is the strength of the field,
  - $d = |\vec{x}_2 - \vec{x}_1|$ is the length of the line segment $\vec{x}_1 \to \vec{x}_2$, and
  - θ is the angle made between the field vector and the line segment.

- Positive charge seeks decreasing electric potential, while negative charge seeks increasing electric potential.
- The electric potential generated by a point particle of charge q' is given by

$$V(r) = \frac{q'}{4\pi\epsilon_o r}$$

  where r is the distance away from the point charge.

- Expressions exist for the electric potential generated by some other common charge distributions (sheets, spherical shells, cylindrical shells).
- Electric potential obeys a superposition principle.

2 Definition of electric potential $V(\vec{x})$

We can assign an *electric potential* $V(\vec{x})$ with any electric field $\vec{E}(\vec{x})$ in the *exactly* the same way we are able to assign a potential energy function $U(\vec{x})$ with any conservative force field $\vec{F}(\vec{x})$. An electric potential $V(\vec{x})$ is something that is *generated* by a collection of charges, just like an electric field $\vec{E}(\vec{x})$.

Recall that if we add a particle of charge q to a point $\vec{x}$ in space where the electric field is $\vec{E}(\vec{x})$, the particle feels a force $\vec{F} = q\vec{E}(\vec{x})$. Likewise, if we add a particle of charge q to a point $\vec{x}$ in space where the electric potential is $V(\vec{x})$, the particle's potential energy[1] is

$$U(\vec{x}) = qV(\vec{x}) \tag{1}$$

This is the definition of the electric potential. Its SI unit is the Volt (abbreviated V), which in light of the previous equation should then be equal to

- the SI unit for energy, which is the Joule (J), *divided by*
- the SI unit for charge, which is the Coulomb (C)

i.e.

1 V = 1 J/C (SI units)

Because of the name of the unit, the electric potential at a point or between two points is often referred to as the *voltage* between those points. Electric potential is thus the electrical quantity most familiar to us. To get a sense of magnitude:

- cell phone charger: 5V
- voltage coming out of wall socket: 120V
- resting potential of a neuron: 70mV
- person before getting shocked by doorknob: 5kV
- fully charged defibrillator: 5kV
- high voltage power lines: 100kV
- thunderstorms: >1GV = 10^6V!!!

[1] Possible point of confusion: as with the case of potential energy $U(\vec{x})$, it is only *differences* in electric potential that have any physical signficance. For example, the points $\vec{x}$ where $V(\vec{x}) = 0$ are only special in that they are at the same electric potential as the fixed point $\vec{x}_o$ we needed to specify in order to define the potential energy function (lecture 7, section 5). Recall from that lecture that we can pick the fixed point any way we like, so we are always free to make the choice most convenient for our situation.

Many concepts we explore later in the semester (e.g. batteries, capacitance, resistance, magnetic inductance) will find their most natural expression in terms of the electric potential $V(\vec{x})$ we have just defined.

As we cover these topics, always keep somewhere in the back of your mind that, when it comes down to it, an electric potential (or voltage) is something you take and multiply by a charge to get an energy. More precisely we would say:

> The potential energy ΔU gained by a particle of charge q going from a point $\vec{x}_1$ to a point $\vec{x}_2$ is given by the particle's charge q multiplied by the electric potential difference $\Delta V = V(\vec{x}_2) - V(\vec{x}_1)$ between those two points. In short,
>
> $$\Delta U = q\Delta V$$

If that is the *only* form of potential energy the particle possesses, then by conservation of energy then we can say that the change ΔKE in the particle's kinetic energy is *opposite*, since $\Delta KE + \Delta U = 0$, i.e.

$$\Delta KE = -q\Delta V$$

As an example, to obtain very clean surfaces (for, e.g., semiconductor manufacturing), a stream of noble gas atoms (neon, argon, xenon, etc.) strike the surface with a high incident velocity, etching away the contaminated layer and exposing clean material underneath. This process is known as "sputtering". To obtain the high velocities necessary for etching, the argon atoms are ionized (i.e. they have an electron removed) some distance above the surface to be cleaned. There is also arranged so that there is a -400V potential difference between the surface to be sputtered and the atom's location when it is ionized. What is the best estimate for the ionized atom's kinetic energy when it strikes the surface? Assume all of its kinetic energy comes from the electric potential. Answer in energy units of J.

The effectiveness of the sputtering process is measured by the "sputter yield", which is the number of surface atoms ejected per incident noble gas ion. The sputter yield for xenon atoms accelerated by a 400V electric potential is plotted in figure 1. Note how the metals we think of as soft – copper, silver, gold – have high yields, while hard materials like carbon (diamond) and titanium have low yields.

3 Electric potential for constant electric fields

From the definition (equation 1) it can be shown that the electric potential difference $V(\vec{x}_2)-V(\vec{x}_1)$ between two points $\vec{x}_1$ and $\vec{x}_2$ in a region of *constant* electric field $\vec{E}_o$ is:

$$V(\vec{x}_2) - V(\vec{x}_1) = -\vec{E}_o \cdot (\vec{x}_2 - \vec{x}_1) = |\vec{E}_o||\vec{x}_2 - \vec{x}_1| \cos\theta \tag{2}$$

where θ is angle made between $\vec{E}_o$ and the vector $\vec{x}_2 - \vec{x}_1$. See figure 2 for a diagram of the variables.

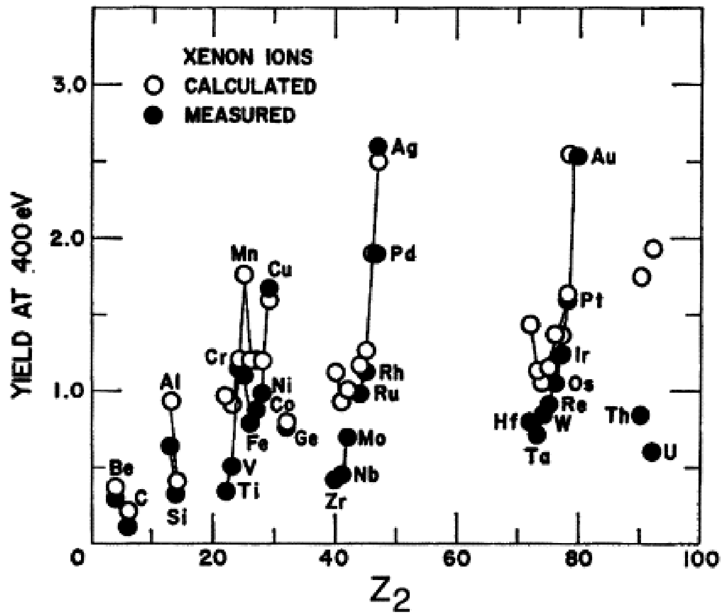


Figure 1: Sputter yields for 400eV xenon ions.

With this formula we can learn something about what the electric potential is telling us in terms of its effect on charged particles. Suppose we are in a region where the electric field is a constant value $\vec{E}_o$. If we pick a point A in this region and another point B so that the segment $\overrightarrow{AB}$ points parallel to $\vec{E}_o$ (see figure 3), then we know from $\vec{F} = q\vec{E}$ that:

- a proton placed at A will be attracted to B, and thus
- an electron placed at B will be attracted to A.

Now if we compare the electric potential at these points using equation (2) we find

$$V_B - V_A = -E_o d_{AB} \tag{3}$$

where $E_o = |\vec{E}_o|$ is the strength of the electric field and d_{AB} is the length of the line segment $\overrightarrow{AB}$. The right hand side is therefore negative quantity, we find that the electric potential at B is thus *lower* than it is at A, i.e.

$$V_B < V_A$$

From this observation we arrive at the general principle that

- positive charge is attracted to regions of lower potential and
- negative charge is attracted to regions of higher potential

Note however, that both positive and negative charges are attracted to regions of lower potential energy. Why is this?

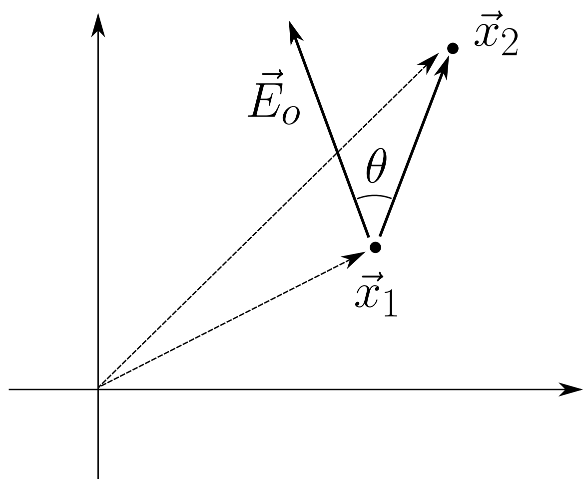


Figure 2: Illustration for the equation (2).

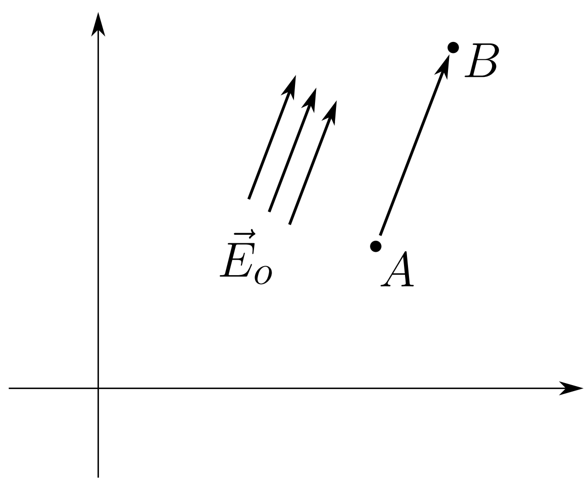


Figure 3: The potential energy decreases as you move "downstream" of an electric field line. See equation (3).

4 Electric potential for different charge distributions

We began the lecture by stating that any electric field $\vec{E}(\vec{x})$ (or, equivalently, any collection of charges) has an associated electric potential $V(\vec{x})$. In lecture 5 we learned how Gauss' law allowed us to determine the electric field generated by a variety of charge distributions (the sphere, the cylinder, the plane). What are the electric potentials associated with these charge distributions? What about the potential generated by a *point charge*, i.e. we know the Coulomb force but what is the Coulomb *potential*?

With the exception of the infinite plane, there are no simple rules to determine the potentials for these distributions since the electric fields they generate are not constant. We state the results along with some comments below.

4.1 Point charge

The electric potential $V(\vec{x})$ at a point $\vec{x}$ generated by a point particle of charge q' located at a position $\vec{x}'$ is

$$V(\vec{x}) = \frac{q'}{4\pi\epsilon_o r} = \frac{kq'}{r} \tag{4}$$

where

$$r \equiv |\vec{x} - \vec{x}'|$$

is the distance between $\vec{x}$ and $\vec{x}'$. A sketch of the r dependence is shown in figure 4. Notice that the electric potential $V(\vec{x})$ at a point $\vec{x}$ is, unlike

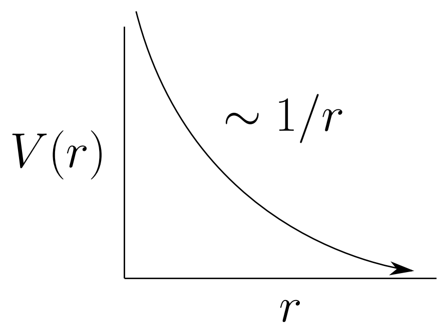


Figure 4: Electric potential of a point particle with charge $q' > 0$. An additional positive charge placed near the original particle would "seek" a lower potential by increasing the separation r, i.e. repelling from the like charge. If $q' < 0$ then the graph is flipped across the x axis.

the electric field $\vec{E}(\vec{x})$, not a vector but just a *number*. This convenience helps simplify a lot of problem solving, though perhaps at the expense of intuition, since we no longer can easily tell what the forces on our particles are.

4.2 Infinite plane

In lecture 7 we determined the potential energy $U(x, y, z)$ for a particle of charge q with coordinates x, y, z when there is an infinite charged plane with surface charge σ lying in the xy plane. We found that

$$U(x, y, z) = -\frac{q\sigma|z|}{2\epsilon_o}$$

From the definition (equation (1)) of the electric potential we divide both sides by q to obtain

$$V(x, y, z) = -\frac{\sigma|z|}{2\epsilon_o} \tag{5}$$

See figure 5 for an illustration.

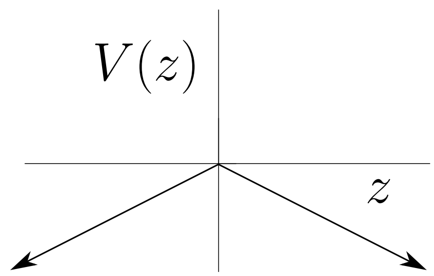


Figure 5: Electric potential generate by infinite sheet of surface charge density $\sigma > 0$. For $\sigma < 0$ we flip the graph over the x-axis.

4.3 Spherical shell

The potential $V(r)$ outside of a thin spherical shell of radius R with a uniformly distributed charge Q is equal to the potential generated by a point charge Q at the shell's center, i.e.

$$V(r) = \frac{kQ}{r},\ r > R \tag{6}$$

where r is the distance to the sphere's center. How about the potential at points $r < R$ *inside* the shell? Well, by Gauss' law the electric field is zero inside the shell (no charge enclosed by Gaussian sphere of radius $r < R$), so by equation (3) the potential *difference* between any two points inside the shell is *zero*[2]! Therefore the potential inside the shell is *constant* and equal to the potential $V(R)$ just outside the shell ($r = R$):

$$V(r) = \frac{kQ}{R},\ r < R \tag{7}$$

see figure 6 for an illustration.

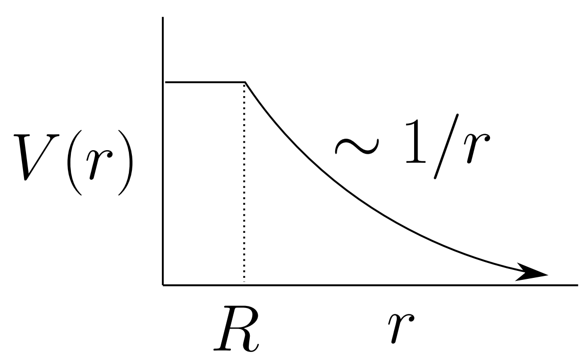


Figure 6: Electric potential generated by a thin spherical shell, shown for $Q > 0$. If $Q < 0$ the graph is flipped across the x axis.

[2] we will use this argument again when we discuss the potential at conductors.

5 Cylindrical shell

An infinitely long cylindrical shell with radius a and linear charge density λ (figure 7) generates a potential $V(r)$ at points $r < a$ outside the shell given by

$$V(r) = -\frac{\lambda}{2\pi\epsilon_o} \ln \frac{r}{a},\ r > a \tag{8}$$

As with the case of the spherical shell, we have zero electric field for points $r < a$ inside the shell and thus a constant electric potential equal to the potential just outside the shell, i.e.

$$V(r) = -\frac{\lambda}{2\pi\epsilon_o} \ln \frac{a}{a} = 0,\ r < a \tag{9}$$

where we used the identity $\ln 1 = 0$. See figure 8 for a sketch of the r dependence.

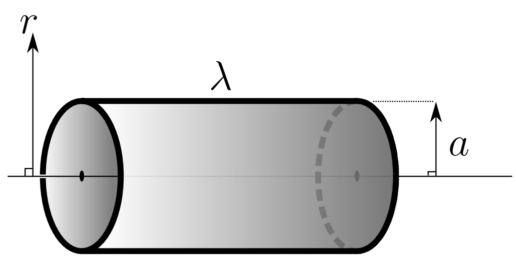


Figure 7: Infinitely long charged cylinder with radius a and linear charge density λ. The coordinate r indicates the distance from the cylinder axis.

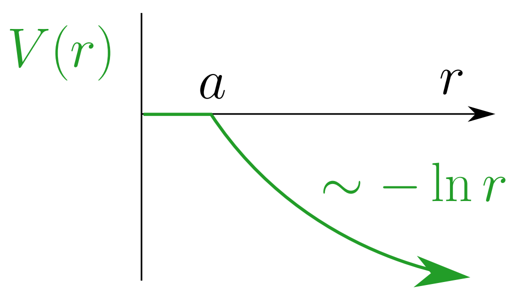


Figure 8: Plot of electric potential of an infinite charged cylinder versus the distance r from the cylinder axis. Graph is shown for a positively charged cylinder $\lambda > 0$. The graph for the case $\lambda < 0$ is obtained by flipping about the horizontal axis.

6 Superposition principle and examples

Electric potentials follow the same superposition principle as electric fields. If a collection of charges A generates an electric potential $V_A(\vec{x})$ and a collection B likewise generates $V_B(\vec{x})$, then the potential $V_{AB}(\vec{x})$ generated by the two distributions collectively is simply their *sum*, i.e.

$$V_{AB}(\vec{x}) = V_A(\vec{x}) + V_B(\vec{x})$$

See figure 9 for an illustration.

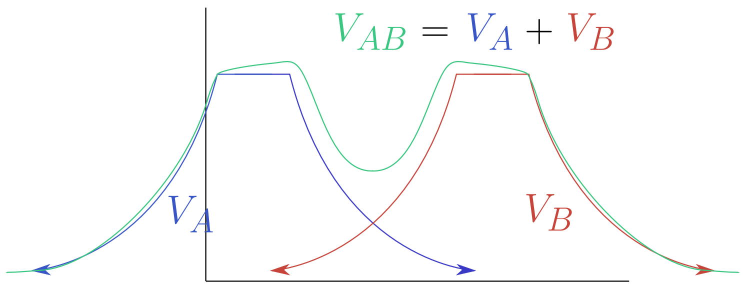
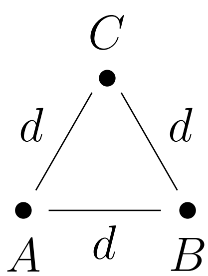


Figure 9: Superposition principle for electric potentials

Example: three point charges



Two particles A and B, both with charge $q' > 0$, are fixed in space and separated by a distance d.

1. How much work does it take to bring a third particle C of charge $q > 0$ from very far away to a point where ABC forms an equilateral triangle (see the above figure)?

2. If we release particle C from rest in this position, what is its kinetic energy at some time in the distant future?

Solution:

- The work W done along a path by a conservative force is given by

  $$W = U_1 - U_2$$

  where U_1 and U_2 are the potential energies at the start and endpoints, respectively. The potential energy U of particle C is given by qV, where V is the electric potential generated by particles A and B. By the superposition principle, the electric potential generated at a point is given by the sum of the electric potentials generated by A and B separately.

  At a point very far away, the electric potential generated by either particle is very small, since the potential generated by a point particle decreases as the inverse of the distance ($\propto 1/r$) from the particle. Therefore,

$$U_1 = qV_1 = q0 = 0$$

At the end point of the path, where ABC forms an equilateral triangle, the electric potential generated by either particle at the location of particle C is given by $\frac{kq'}{r}$. Adding the two potentials together we find

$$U_2 = qV_2 = q\left(\frac{kq'}{r} + \frac{kq'}{r}\right) = 2\frac{qq'}{r}$$

So that the work done along the path is

$$W = U_1 - U_2 = -2\frac{qq'}{r}$$

- Since all the charges are assumed positive, if particle C is released it will be repelled by particles A and B so that at some much later time it will be very far away from them. Therefore, the trajectory particle C travels is simply the reverse of the path considered in the first question. The work W' done along the trajectory is then simply the negative of the work W done along the original path. By the work-energy principle, the kinetic energy ΔKE gained by a particle along a trajectory is equal to the work done along the trajectory. Since the initial kinetic energy KE_o of particle C is presumed to be zero, the final kinetic energy KE' of C is

$$KE' = KE_o + \Delta KE = 0 - W = 2\frac{qq'}{r}$$

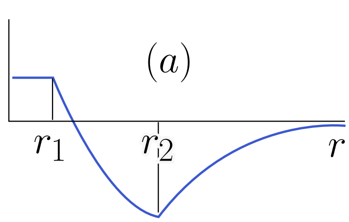
Instapoll questions:

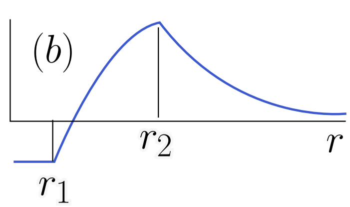
1. Two charged particles are placed on a line. One particle, of charge $q > 0$, sits at position $d > 0$ and another particle, of charge $-q$, sits at a position $-d$. Draw a picture!

   (a) What is the electric potential at the origin?

   (b) *With this in mind*, what is the direction of the electric field at the origin?

2. Consider two concentric spherical shells, one of radius r_1 with charge $Q > 0$, and another with a radius $r_2 > r_1$ with a charge $-2Q$. Which of these graphs best represent the electric potential generated by this charge distribution? *Hint: what do we expect for distances $r > r_2$?*





Solutions:

1. (a) According to the superposition principle, the electric potential V generated by the pair of charges at the origin is equal the sum $V_R + V_L$ of the potentials generated by the right and left charges individually. Applying equation (4) to the right particle we make the following substitutions:

   - $q' \to q$
   - $r \to d$

   since the particle on the left has a charge q and is a distance d from the origin. Therefore:

$$V_R = \frac{kq}{d}$$

   and likewise for the left particle

   - $q' \to -q$
   - $r \to d$

   So that

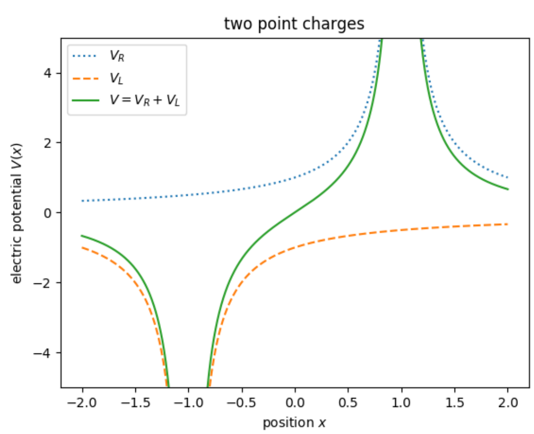
$$V_L = \frac{k(-q)}{d} = -\frac{kq}{d} = -V_R$$

   Adding them together we get

$$V = V_R + V_L = 0$$

   (b) The electric field points to the left. This can be seen in two ways:

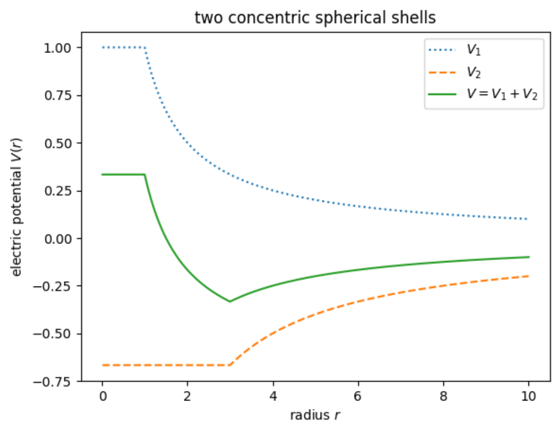
   - A proton placed at the center will experience an acceleration to the left due to both the attraction towards the negatively charged particle to its left and repulsion from the positively charged particle to its right. Since the electric field points parallel to the acceleration of a proton, we conclude the electric field points to the left.
   - Look at a plot of the electric potential generated by the two charges at an arbitrary position along the line joining them:



Observe that the potential *decreases* as the we move to the left of the origin. Since protons seek regions of decreasing electric potential, we reach the same conclusion as previous: a proton is attracted to the left, and thus the electric field points to the left.

This example points out that the electric field can be nonzero at point where the electric potential is zero. This should not be surprising since it is only *differences* in electric potential which have physical significance.

2. Applying equations (6) and (7) we obtain the following plot:



where have set $r_1 = 1$ and $r_2 = 3$. So it appears that case (a) is the best match.

How could we see this by just looking at the graphs? For points $r > r_2$, both spheres generate potentials equivalent to the potentials generated by point particles located at the spheres' center with the same total charge as the spheres. In other words, outside of the second sphere, the potential is given by the potential generated by a particle of charge $q' = 1 + (-2) = -1 < 0$. Plot (a) shows the total potential $V(x)$ gradually increasing with increasing $r > r_2$, as we expect for the potential due to a negative point charge.

7 Field, force, energy, potential

This section may be helpful if you are having difficulty keeping straight the four concepts of electric field $\vec{E}$, force $\vec{F}$, potential energy U, and electric potential V. It is only intended as a rough conceptual summary.

In my mind the main characters in electrostatics can be conveniently arranged in a square. See figure 10. Let's take a walk around the square:

- *left to right*: Electric fields $\vec{E}(\vec{x})$ and electric potentials $V(\vec{x})$ are *generated* by collections of charge that occupy space. If we add into this space *another* particle of charge q at a point $\vec{x}$, then
  - the force on this new particle is given by $\vec{F} = q\vec{E}(\vec{x})$, and likewise

$$
\begin{tikzcd}
\vec{E} \arrow[r] \arrow[d] & \vec{F} \arrow[d] \\
V \arrow[r] & U
\end{tikzcd}
$$

Figure 10: Electrostatics square

  - the potential energy of this new particle is given by $U(\vec{x}) = qV(\vec{x})$

  Force and potential energy are in a sense more "real" because they have a direct impact on the *motion* of particles. The force impacts motion through Newton's law $F = ma$, and the potential energy impacts motion through energy conservation $KE_o + U_o = KE' + U'$. The electric field and electric potential are convenient conceptual tools, but are less real in the sense that they only contain information on what *would* happen if an additional particle were added to the system.

- *top to bottom*:

  - Potential energy U is obtained by acting a force F over a distance d, so that roughly speaking

    $$U \sim Fd$$

  - Likewise, electric potential V is obtained by acting an electric field E over a distance d, or schematically

    $$V \sim Ed$$

    See equation (3) for the precise relationship.