302L S20 lecture 7 - electric potential energy

1 Summary

- Work (denoted by W) is a quantity defined for a force field $\vec{F}(\vec{x})$ acting in some space and a curve C in that space. For a constant force field $\vec{F}(\vec{x}) = \vec{F}_o$ and a curve C given by a straight line from a point $\vec{x}_1$ to another point $\vec{x}_2$ we have

$$W = \vec{F}_o \cdot (\vec{x}_2 - \vec{x}_1)$$

  for a general force field $\vec{F}(\vec{x})$ and arbitrary curve C the procedure for determining W is more complicated.

- For certain force fields $\vec{F}(\vec{x})$, known as *conservative force fields*, the work depends only on the start and end points of a curve. Many force fields in nature are conservative (constant force, gravitational force, Coulomb force).

- We can associate with any conservative force field $\vec{F}(\vec{x})$ a *potential energy function* $U(\vec{x})$. Differences in the potential energy function measure the work needed to travel between two points. The potential energy function $U(x)$ obeys a superposition principle.

- The potential energy function associated with an infinite charged sheet is proportional to the distance from the sheet.

- The *total energy* E of a particle (not to be confused with *electric field* $\vec{E}(\vec{x})$!) in a conservative force field at some instant is defined as the sum $E = T + U$ of the particle's kinetic energy T and its potential energy U.

- The total energy E of a particle is conserved, i.e. its value does not change as time progresses.

2 Force fields

Consider a particle under the influence of a force field $\vec{F}(\vec{x})$, so that when the particle is at a position $\vec{x}$ the force on the particle is $\vec{F}(\vec{x})$.[1] We list some important examples:

[1] A force *field* is then, like the electric *field*, a *function* taking in a point in space and returning *vector* $\vec{F}(\vec{x})$.

- In the case of a free particle, the force is simply zero at all points in space.
- The next simplest example is the case of a *constant* force field, where at any point in space the particle experiences the same force vector $\vec{F}$.
  - The most familiar example is the force of gravity at the earth's surface. If we choose a coordinate system where the z direction points straight up and the x and y directions run parallel to the earth's surface, then a particle of mass m at any point near the earth's surface experiences a downwards gravitational force $\vec{F} = -mg\hat{\mathbf{k}}$, where $g \approx 9.8\text{m/s}^2$.
  - In lecture 5 we found that an infinite sheet with uniform surface charge density σ generates an electric field everywhere above the surface equal to $\vec{E} = \frac{\sigma}{2\epsilon_o}\hat{n}$, where $\hat{n}$ is the direction normal to the surface. A particle of charge q would then experience a *constant* force field $\vec{F} = \frac{q\sigma}{\epsilon_o}\hat{n}$. Note that the force field is not totally constant in that it reverses direction when the particle is *below* the sheet.
  - In an important sense we can always, for any *smoothly* varying force field, find a small enough region where the force field is essentially constant. This is in fact the situation for gravity at the earth's surface. We know that the strength of the gravitational force on two orbiting bodies (e.g. the earth and sun, or moon and earth) varies as the inverse square of the distance between them. The direction of the force on an orbiting object also varies with the object's position, staying always parallel to the line joining the two oribiting bodies.

    This is also true for the gravitational force between a projectile and the earth, so strictly speaking the projectile's force field is not constant but varies in strength and direction as the projectile travels. However, for any reasonable trajectory (i.e. excluding rocket launches), the vector joining the earth's center of mass to the projectile is *essentially* constant over the full range of the projectile's motion. Therefore, the force exerted by the earth on the projectile is also essentially constant.

    For a very similar reason we can also say that the electric field in the region above the center of a flat plate of charge density σ and *finite* (i.e. non-infinite) area is essentially constant and equal to the value $\vec{E} = \frac{\sigma}{2\epsilon_o}\hat{n}$ calculated for the infinite charged plate. Strictly speaking, the true electric field would be equal to the sum of (see figure 1 for an illustration):
    * the field $\vec{E}_+(\vec{x})$ generated by an infinite plate P_+ of surface charge density σ, and
    * the field $\vec{E}_-(\vec{x})$ generated by an second infinite plate P_- of surface charge density $-\sigma$ that has a hole where actual plate lies.

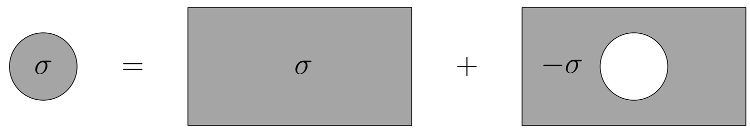

Figure 1: Illustration of finite-area charged plate as sum of two infinite area plates of opposite charge, where the second plate contains a void where the plate originally lied.

This second field $\vec{E}_{-}(\vec{x})$, however, will be essentially zero at points $\vec{x}$ in the region just above the center of the plate, since the charges comprising plate P_{-} will exert forces primarily parallel to the plate and these forces will cancel. We are left then with the electric field $\vec{E}_{+}(\vec{x})$ generated by P_{+}, which is given by $\vec{E}_{+}(\vec{x}) = \frac{\sigma}{2\epsilon_o}\hat{n}$.

- We also have force fields that vary by an amount proportional to some displacement x, i.e. we have objects obeying “Hook’s Law”:

  $$F(x) = -x$$

  Where k, known as the “spring constant”, does not depend on the displacement x. Objects under the influence of this force field will oscillate about their equilibrium position $x = 0$.

- Finally we include the example of force fields obeying an inverse square law which is obeyed by orbiting bodies (Newton’s law of universal gravitation) and charged partices (Coulomb’s law). We described this force field earlier in its connection to projectile motion.

3 Work

Given a force field $\vec{F}(\vec{x})$ and some curve C in space we can define a quantity termed the *work* (denoted by the symbol W) done by the force along the curve. The work is defined explicitly for the case of a *constant* force field $\vec{F}(\vec{x}) = \vec{F}_o$ and a curve C given by a straight line from a point $\vec{x}_1$ to another point $\vec{x}_2$. Namely:

$$W = \vec{F}_o \cdot (\vec{x}_2 - \vec{x}_1) \tag{1}$$

To obtain an approximate expression for the work done by a general force field $\vec{F}(\vec{x})$ along a general curve C using the following the procedure:

1. Approximate the curve C by a bunch of small line segments $L_1, L_2, \ldots, L_N$ joined end to end. The line segments should be small enough that the force $\vec{F}(\vec{x})$ is approximately equal for each point $\vec{x}$ along a line segment.

2. Calculate the approximate work

   $$W_i \equiv \vec{F}\left(\bar{\vec{x}}_i\right) \cdot \left(\vec{x}_i^{+} - \vec{x}_i^{-}\right)$$

   done along a line segment L_i, where $\vec{x}_i^{-}$, $\bar{\vec{x}}_i$, and $\vec{x}_i^{+}$ are the beginning, middle, and end points of theline segment L_i.

3. The total work W is the the sum $\sum_{i=1}^{N} W_i$ of the works done along each line segment L_i.

In the homework you'll apply this procedure to calculate the (exact!) work done in extending a spring from its equilibrium position $x = 0$ to some displacement $x = A$. You will find that

$$W = -\frac{1}{2}kA^2$$

4 Conservative Force Fields

Though it is not obvious, it turns out that for many force fields, including all the ones described earlier, the work done along some curve depends only on the start and end points of the curve and not on the details in between. Force fields having this property are known as *conservative fields*.

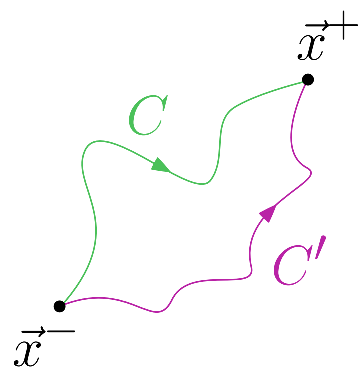

Figure 2: Diagram illustrating path independence.

Figure 2 illustrates this concept. If the space in the figure is under the influence of a conservative force field, we have then that the work for curves C and C' are equal:

$$W_C = W_{C'}$$

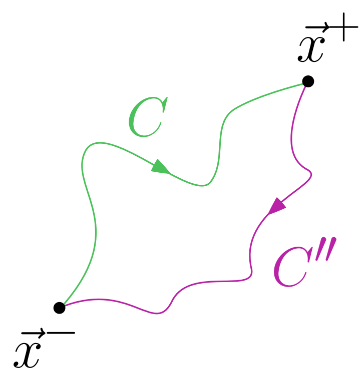

Figure 3: A closed path obtained by joining C and C''.

Refer now to figure 3. Here we have switch the endpoints of the curve C' around, obtaining a new curve C''. By the definition of work I claim that

$$W_{C''} = -W_{C'}$$

(Convince yourself that this is true.) I also note that a *closed path* can be obtained by joining curves C and C''. The work W done along this closed path is equal to the sum of the works W_C and $W_{C''}$, giving

$$W = W_C + W_{C''} = W_C - W_{C'} = W_C - W_C = 0$$

So that the work done along a closed curve by a conservative vector field is always zero.

5 Potential Energy

For a conservative force field $\vec{F}(\vec{x})$ we can define an associated *potential energy function* $U(\vec{x})$ which is given by

$$U(\vec{x}) = W_{\vec{x}\to\vec{x}_o}$$

where $W_{\vec{x}\to\vec{x}_o}$ is the work done by the force field along a curve going from $\vec{x}$ to some *fixed point* $\vec{x}_o$. Since the force field is conservative it does not matter which curve we draw so long as it starts at $\vec{x}$ and ends at $\vec{x}_o$.

By drawing a closed curve going

1. from $\vec{x_o}$ to some point $\vec{x_1}$, then
2. from $\vec{x_1}$ to another point $\vec{x_2}$, then
3. from $\vec{x_2}$ back to $\vec{x_0}$,

we find that the *difference* in potential energies $U(\vec{x}_2) - U(\vec{x}_1)$ is equal to the work $W_{\vec{x}_2\to\vec{x}_1}$ done along a curve going from $\vec{x}_2$ to $\vec{x_1}$ and is thus independent of which point we choose as our fixed point $\vec{x}_o$. This highlights the important point that it is only *differences* in potential energy that have physical significance.

The potential energy function obeys the superposition principle in that if a conservative force field $\vec{F}(\vec{x})$ can be written as the sum of two conservative force fields $\vec{F}_1(\vec{x}) + \vec{F}_2(\vec{x})$, then the potential energy function $U(\vec{x})$ associated with $\vec{F}(\vec{x})$ can be written as

$$U(\vec{x}) = U_1(\vec{x}) + U_2(\vec{x})$$

where $U_1(\vec{x})$ and $U_2(\vec{x})$ are the potential energy functions associated with $\vec{F}_1(\vec{x})$ and $\vec{F}_2(\vec{x})$, respectively.

Let's work out the potential energy function $U(\vec{x})$ associated with a constant force field $\vec{F}(\vec{x}) = \vec{F}_o$. For convenience let's select our fixed point to be the origin $\vec{0} \equiv (0,0,0)$. The potential energy $U(\vec{x})$ is then the work along a curve starting at $\vec{x}$ and ending at $\vec{0}$. We can pick any curve we like, but we will pick the simplest one – the straight line segment. Using equation (1) we find

$$U(\vec{x}) = \vec{F_o} \cdot \left(\vec{0} - \vec{x}\right) = -\vec{F_o} \cdot \vec{x}$$

Taking $\vec{F_o} = -mg\hat{\mathbf{k}}$ and letting $\vec{x} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ we get for the gravitational potential energy at the earth's surface a function

$$U(x, y, z) = -\left(\left(-mg\hat{\mathbf{k}}\right) \cdot \left(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}\right)\right) = mgz$$

Similarly, for a particle of charge q above an infinite sheet of surface charge density σ with a normal vector $\hat{n} = \hat{\mathbf{k}}$ we have

$$U(x, y, z) = -\frac{q\sigma z}{2\epsilon_o} \tag{2}$$

where we have set the plane $z = 0$ (i.e. the xy plane) to coincide with the charged sheet.

Instapoll question: What is the potential energy *below* the charged sheet (i.e. for $z < 0$)?

1. $U(x, y, z) = -\frac{q\sigma z}{2\epsilon_o}$
2. $U(x, y, z) = +\frac{q\sigma z}{2\epsilon_o}$
3. $U(x, y, z) = 0$

Hint: how does the electric field above the plate differ from the electric field below the plate?

Answer: The electric field below the field points *down*, opposite to its direction above the sheet. This makes sense since, no matter whether you're a proton above or below the sheet, you will still be repelled by the sheet if σ is positive (and vice versa). Therefore, the work done taking a charge to the sheet depends only on the distance the charge is from the sheet. We can write a single equation that handles both the $z > 0$ and $z < 0$ cases by writing

$$U(z) = -\frac{q\sigma|z|}{2\epsilon_o}$$

Now suppose we add a second sheet of charge, parallel to the first sheet, with an *opposite* surface charge density $-\sigma$, spaced a distance d away from the first sheet. What is the resulting potential energy function $U(x)$ between the two sheets? See figure 4 for an illustration.

By the superposition principle the potential energy function of the combined system $U(x, y, z)$ should be equal to the sum $U_+(x, y, z) + U_-(x, y, z)$ of

- the potential $U_+(x, y, z) = -\frac{q\sigma z}{2\epsilon_o}$ of the lower sheet with $+\sigma$ surface density, which we already solved for, plus
- the potential $U_-(x, y, z)$ of the upper sheet with $-\sigma$ surface charge density.

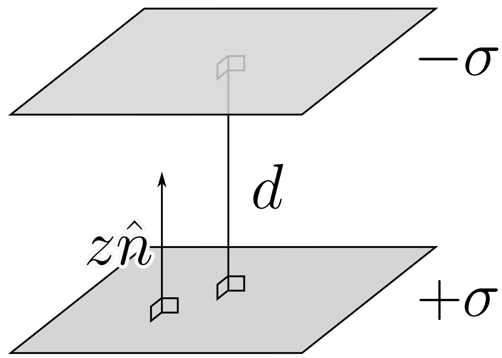

Figure 4: Two parallel planes with opposite surface charge densities $\pm\sigma$.

However, the electric field generated by the lower sheet in the region below it is exactly the same as the electric field generated by the upper sheet in the region above it! Therefore, the potential energy function in the region *between* the sheets is simply *double* what it was before we added the second sheet, i.e. in the region $0 < z < d$ we have

$$U(x, y, z) = U_+(x, y, z) + U_-(x, y, z) = -\frac{q\sigma z}{\epsilon_o}$$

To think about at home: what is the potential below the lower sheet now? Above the upper sheet?

As a bit of an aside I should mention that the potential energy actually encodes all the information about its associated (conservative) force field, so that we can recover the force field from knowledge of its associated potential energy function. For instance, to get the $\hat{\mathbf{i}}$ component at a point $\vec{x}$ of the force field $\vec{F}(\vec{x})$ associated with a potential energy function $U(\vec{x})$, construct a line segment L with starting point $\vec{x}$ and endpoint $\vec{x}+\Delta x\hat{\mathbf{i}}$. Make Δx small enough that the force field is essentially constant along the line segment. We find then that work along the line segment is

$$W_L = \vec{F}(\vec{x}) \cdot (\vec{x} + \Delta x\hat{\mathbf{i}} - \vec{x}) = \Delta x\vec{F}(\vec{x}) \cdot \hat{\mathbf{i}} = U(\vec{x}) - U(\vec{x} - \Delta x\hat{\mathbf{i}})$$

or in other words

$$\vec{F}(\vec{x}) \cdot \hat{\mathbf{i}} = \frac{U(\vec{x}) - U(\vec{x} - \Delta x\hat{\mathbf{i}})}{\Delta x}$$

To get the components of the force field along the $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ directions we repeat the procedure, making the substitution $\hat{\mathbf{i}} \to \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} \to \hat{\mathbf{k}}$. By this process we can use the potential energy function as a blood hound to "sniff out" the associated force field.

6 Work-energy Principle

As it stands we haven't given any reason why we should *care* about the work associated with some curve in a force field. The work concept only obtains its significance via the *work-energy principle*, explained below.

Consider a particle moving along a trajectory C under the influence of a force field $\vec{F}(\vec{x})$. Note that a trajectory is not any old curve, but is one where at all points $\vec{x}$ along the trajectory we satisfy Newton's second law so that the particle's acceleration $\vec{a}$ at $\vec{x}$ is related to the force $\vec{F}(\vec{x})$ at $\vec{x}$ by

$$\vec{F}(\vec{x}) = m\vec{a}$$

Since we have a force field $\vec{F}(\vec{x})$ and a curve given by our trajectory C, we can ask about the work W done by the force field along the trajectory. It turns out that, for any force field (conservative or not), the work W along a particle's trajectory is equal to the difference $\Delta T \equiv T' - T_o$ in the particle's initial kinetic energy T_o and its final kinetic energy T', i.e.

$$W = \Delta T = T' - T_o \tag{3}$$

7 Energy conservation

If a particle moving under the influence of a conservative vector field has some trajectory starting at a point $\vec{x}_o$ and ending at a point $\vec{x}'$, then the work-energy principle (equation 3) tells us

$$U_o - U' = W = \Delta T = T' - T_o$$

where $U_o \equiv U(\vec{x}_o)$ and $U' \equiv U(\vec{x}')$. Rearranging we find

$$T_o + U_o = T' + U' \tag{4}$$

Equation (4) tells us that the sum $E \equiv T + U$ of the potential and kinetic energies is *conserved*, i.e. that a loss/gain in either of the quantities over time implies a corresponding gain/loss in the other. We call this sum E the *total energy* (or just the *energy*) of the system. Conservation of energy is great convenience, aiding in the solution of many varieties of physics problems.

Energy conservation is also in some sense surprising. Refer to figure 5, which shows three season-appropriate projectile trajectories. All three trajectories have the same start and end points. Despite the large differences in the magnitudes/directions of their initial velocity vectors, we find the difference in their final and initial kinetic energies is the same.

As an example, consider an electron located just outside an infinite sheet of postive surface charge density $\sigma > 0$. It is initially traveling with kinetic energy T_o in the direction normal to the plate. How far does the electron get from the plate before it turns around and heads back? Assume that gravity can be neglected, i.e. $g \to 0$.

Answer:

From earlier we saw that the potential energy $U(z)$ of a particle with charge q when it is a distance z away from an infinite sheet with surface charge density σ is

$$U(z) = -\frac{q\sigma}{2\epsilon_o}z$$

in our case our particle is an electron (i.e. $q = -e$)so

$$U(z) = \frac{e\sigma}{2\epsilon_o}z$$

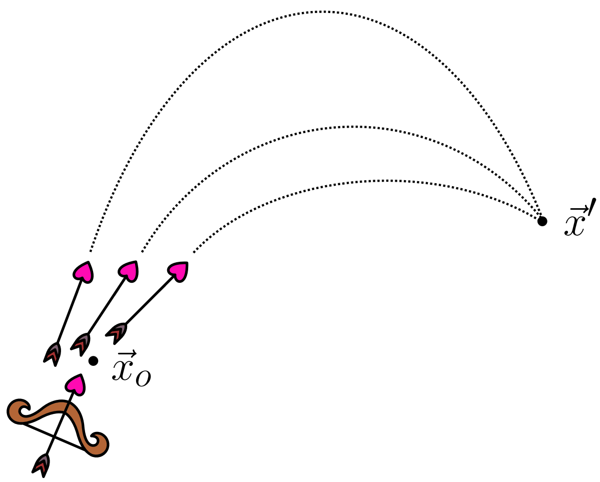

Figure 5: Diagram of three different trajectories with identical starting and ending points.

The total energy E of the electron is given by the sum of its kinetic and potential energies. Since the electron is initially at $z = 0$ we can say that the particle starts with a total energy E given by the sum

$$E = T_o + U(z = 0) = T_o + 0 = T_o$$

Conservation of energy tells us that if, sometime later, the electron is now at a distance z from the surface, we can determine its new kinetic energy T' by

$$T' = E - U(z) = T_o - U(z)$$

The particle begins moving straight away from the surface, so that all of its kinetic energy is from its velocity in the z direction. When the particle turns around then its total velocity is exactly zero, and therefore we want to know the distance z when

$$T' = T_o - U(z) = 0$$

or

$$U(z) = \frac{e\sigma}{2\epsilon_o} z = T_o$$

or, after some algebra,

$$z = \frac{2T_o\epsilon_o}{e\sigma}$$