Lecture 5: Applications of Gauss' Law

1 Gauss' Law Recap

In the previous lecture we introduced Gauss' law, which states that the electric flux Φ through a closed surface is given by

$$\Phi = q_{enc}/\epsilon_o$$

where q_{enc} is the charge enclosed by the surface, and $\epsilon_o \approx 8.85 \cdot 10^{-12}\,\mathrm{C}^2\ \mathrm{N}^{-1}\ \mathrm{m}^{-2}$ is the *permittivity of free space*, a constant introduced from Coulomb's law.

We also determined an expression for the electric flux $\Delta\Phi$ through some small flat patch characterized by the vector $\Delta\vec{A}(\vec{x})$ where:

- $|\Delta\vec{A}| \equiv \Delta A$ is the area of the patch,
- $\Delta\vec{A}/\Delta A$ is a unit vector pointing in the direction *normal* to the patch; that is, the direction perpendicular to any line contained on the patch, and
- $\vec{x}$ is the location of the center of the patch. It is not important where exactly we call the center since the patch is assumed to be very small.

This flux $\Delta\Phi$ we found to be

$$\Delta\Phi = \vec{E}(\vec{x}) \cdot \Delta\vec{A}$$

where $\vec{E}(\vec{x})$ is the electric field vector at the patch center $\vec{x}$. Note that this expression is only perfectly accurate when the electric field is *constant* over the whole patch. Because the patch is assumed to be small the assumption of constancy is a good one. In the case where this assumption is not good we must replace the right hand side with the *average* of the quantity $\vec{E}(\vec{x}') \cdot \Delta\vec{A}$ over every point $\vec{x}'$ on the patch (not just the patch center $\vec{x}$).

If we construct a closed surface by piecing together some large number of small patches $\Delta\vec{A}_i(\vec{x}_i)$, $i = 1, 2, \ldots, N$, then by summing the through through each patch we obtain the following relationship:

$$\sum_{i=1}^{N} \vec{E}(\vec{x}_i) \cdot \Delta\vec{A}_i = q_{enc}/\epsilon_o$$

2 Symmetry

In the sections following we will apply the above formula to determine the electric field generated by continuous charge distributions. In previous lectures and homework we saw that, from the formula for the electric field generated by a point charge and the principle of superposition, we could determine the electric field generated by multiple point charges. This technique however becomes impractical (im*possible*, really) when we have a *continuous* charge distribution, which can be thought of as an *infinite* collection of small charges spread out over space.

In some cases we can get around this difficulty if the distribution possesses a sufficiently high degree of *symmetry*. The symmetry of an object is the number of ways we can transform space and leave the object the same. One simple example of symmetric objects is given in figure 1. For both a butterfly and an isoceles triangle we find that we can reflect each point about some axis and the resulting object will be identical to the object we had before the reflection.

We similarly can construct charge distributions that possess this same symmetry. For instance, referring to the isosceles triangle in figure 1 we can place two equal charges on the two lower corners of the triangle and one unequal charge on the upper corner. After the reflection we obtain an identical charge distribution since the lower charges swap and the upper charge stays put.

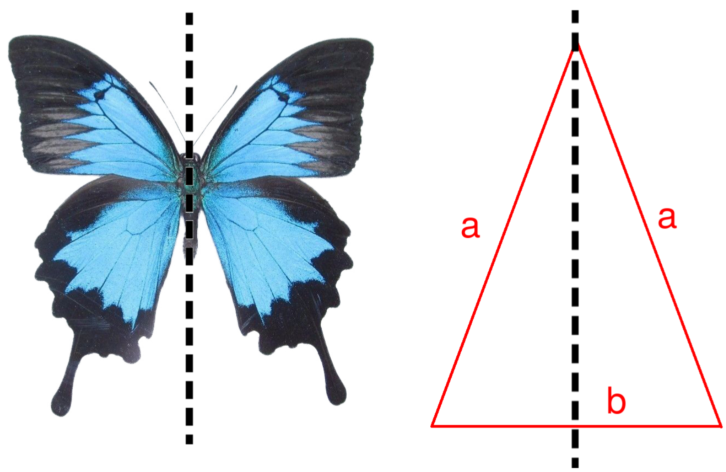

Figure 1: Two objects possessing a single reflection symmetry.

An example of an object with a higher symmetry is given by the equilateral triangle, shown in figure 2. This object now has two more reflection axes in addition to the one possessed by the isosceles triangle (shown in blue). Additionally we have two rotational symmetries (shown in red) in that we can rotate every point about the center of the triangle by $\pm 120°$ and we get the same equilateral triangle back. To obtain a charge distribution with these same symmetries we must place equal charges on all three corners. If the upper charge is not equal to the two lower charges, then application of any of the new symmetry operations will modify the original charge distribution.

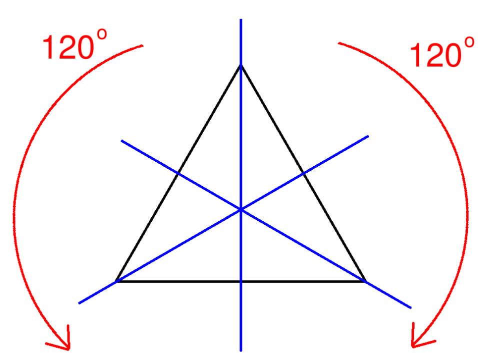

Figure 2: An equilateral triangle.

The circle is the most symmetrical object we can draw on a plane. See figure 3. We see that the circle is *infinitely* symmetric in that we can

- rotate space by *any* angle θ about the circle's center, or
- reflect space about *any* line intersecting the center

and get the same circle back. To obtain a charge distribution with this symmetry it is now necessary to

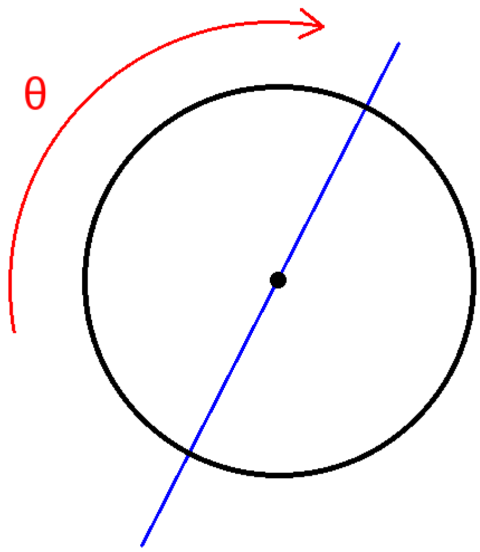

Figure 3: Symmetries of the circle

smear charge uniformly around the entire circle.

3 Applications to various charge distributions

3.1 Spherical distributions

In three dimensions we encounter objects with even higher symmetry than the circle. Our first example will be the spherical charge distribution. See figure 4. With the sphere we can take any line intersecting the sphere's center and rotate space about this line by any angle without modifying the shape of the sphere.

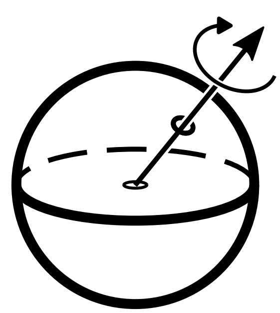

Figure 4: Symmetry of the sphere

Our task is the determine the electric field $\vec{E}(\vec{x})$ generated at a point $\vec{x}$ by a spherical charge distribution with net charge Q. The "general procedure", which we will apply in all the examples covered in this lecture, is the following:

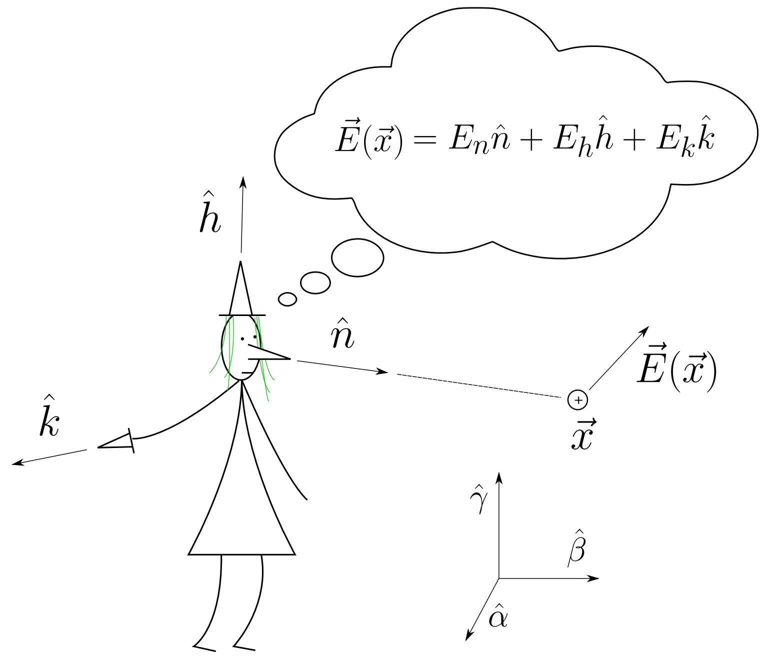

Figure 5: An electro-witch.

1. (a) Exploit the symmetry of the charge distribution to determine which points in space have the same electric field *strength* $|\vec{E}(\vec{x})| \equiv E(\vec{x})$.

   (b) Also using symmetry arguments, determine the *direction* of the electric field $\vec{E}(\vec{x})$ at each point $\vec{x}$.

2. Use the Gauss' law, along with the information obtained in the previous step, to determine the *strength* $|\vec{E}(\vec{x})| \equiv E(\vec{x})$ of the charge distribution at each point $\vec{x}$.

Putting the two steps together we obtain the complete description of the electric field $\vec{E}(\vec{x})$ generated by the charge distribution.

To support us in this endeavor the University has generously allocated some funds which we have used to hire an outside consultancy of "electro-witches". These witches are able to determine the electric field $\vec{E}(\vec{x})$ generated by our charge distribution at any point $\vec{x}$ in space. Refer to figure 5. The witch reports her result as a sequence of three numbers, E_n, E_h, E_k. These numbers correspond to the component of the electric field along three vectors $\hat{n}, \hat{h}, \hat{k}$ so that we have

$$\vec{E}(\vec{x}) = E_n\hat{n} + E_h\hat{h} + E_k\hat{k} \tag{1}$$

The three vectors $\hat{n}, \hat{h}, \hat{k}$ are defined by the following conventions:

- $\hat{n}$ points along her nose, which she always keeps pointed towards the point $\vec{x}$ where she is evaluating the electric field,
- $\hat{h}$ points along the direction of her pointy hat, and
- $\hat{k}$ points along the direction of her knife, which she always keeps pointing out to her right side.

These are vectors are of unit length, i.e.

$$\hat{n} \cdot \hat{n} = \hat{h} \cdot \hat{h} = \hat{k} \cdot \hat{k} = 1$$

and *mutually perpendicular* in the sense that

$$\hat{n} \cdot \hat{h} = \hat{h} \cdot \hat{k} = \hat{k} \cdot \hat{n} = 0$$

Such a trio of vectors are known collectively as an *ortho-normal basis* (see the vector practice assignment).

Note that the direction of these unit vectors are not fixed in space but change as the witch changes her orientation. [1]

We might wonder *how* it is these witches are able to measure the electric field. This information is, of course, closely guarded domain knowledge, but I suspect they do something similar to the following:

- The witch places a proton at the location $\vec{x}$, then
- releases it and measures the different components a_n, a_h, a_k of its acceleration $\vec{a}$ resulting from the forces applied on it by the charge distribution under consideration.
- Then, since in general we have $\vec{F} = m\vec{a} = q\vec{E}$, the witch uses her knowledge of the proton mass $m = m_p$ and charge $q = +e$ to obtain $\vec{E} = \frac{m\vec{a}}{e}$.

Let's assume this to be the case, so that instead of imagining the electric field vector $\vec{E}$ that a witch measures at some point $\vec{x}$ we can equivalently consider the acceleration $\vec{a}$ of a proton released at the point $\vec{x}$. In the diagrams I put a small circle with a plus (+) sign at the points $\vec{x}$ where electric fields are being measured to indicate this hypothetical proton.

Alright then, let's place two electro-witches in the vicinity of our spherical charge distribution, one observing the electric field at a point $\vec{x}_1$, and the other observing the electric field at a point $\vec{x}_2$ (see figure 6). Both points lie a distance r away from the charged sphere's center. For convenience we have set the origin of our external coordinate system (***not*** the witch's internal $\hat{n}, \hat{h}, \hat{k}$ system!) to be the sphere's center, so that $\vec{x}_1$ is then the vector pointing from the sphere's center to the location where we are measuring the electric field (verify this for $\vec{x}_1$ and $\vec{x}_2$ on the figure).

For the spherical case we establish now a rule for the witches requiring that their nose be colinear with the line joining the sphere center to the point $\vec{x}$ where they are currently measuring the electric field. This requirement is in addition to the one stated previously where the nose must also *point towards* the point $\vec{x}$. Combining the two, we can conclude that $\hat{n}$ is *parallel* to $\vec{x}$.

Since the two witches are observing the electric field at two different points $\vec{x}_1$ and $\vec{x}_2$ and with different orientations $\left(\hat{n}_1, \hat{h}_1, \hat{k}_1\right)$ and $\left(\hat{n}_2, \hat{h}_2, \hat{k}_2\right)$, we would expect for an *arbitrary* (i.e. *non*-spherical) charge

[1] In the vector practice homework I emphasize that a vector is a sequence of three numbers. At the risk of being pedantic I will point out that, as it stands, it does not even appear we've *defined* the electric field vector $\vec{E}(\vec{x})$ at a point $\vec{x}$ since we haven't specified how it corresonds to a sequence of three numbers. We do not want to use the three numbers E_n, E_h, E_k since they will change as the orientation of the witch changes and we would like the vector $\vec{E}$ to have a definition independent of the particular way we've oriented the witch. Instead we express the vectors $\hat{n}, \hat{h}, \hat{k}$ in terms of some fixed *external* ortho-normal basis of vectors, which I've labeled $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ (see figure 5 for their illustration). What I mean is that, given a witch orientation, we find the three numbers $n_\alpha, n_\beta, n_\gamma$ that satisfy the equation

$$\hat{n} = n_\alpha \hat{\alpha} + n_\beta \hat{\beta} + n_\gamma \hat{\gamma}$$

and likewise for $\hat{h}$ and $\hat{k}$ so that

$$\hat{h} = h_\alpha \hat{\alpha} + h_\beta \hat{\beta} + h_\gamma \hat{\gamma}$$

and

$$\hat{k} = k_\alpha \hat{\alpha} + k_\beta \hat{\beta} + k_\gamma \hat{\gamma}$$

It doesn't matter how we pick $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ so long as they are fixed (i.e. independent of any witch's orientation) and ortho-normal. We then *define* $\hat{n}$ to be the sequence of coefficients $(n_\alpha, n_\beta, n_\gamma)$, and similarly for $\hat{h}$ and $\hat{k}$. Since the $\hat{n}, \hat{h}, \hat{k}$ are now each associated with their own triplet of numbers, we can then use equation 1 to associate each electric field vector $\vec{E}(\vec{x})$ with its own triplet of numbers that is independent of the orientation of the witch used to measure it.

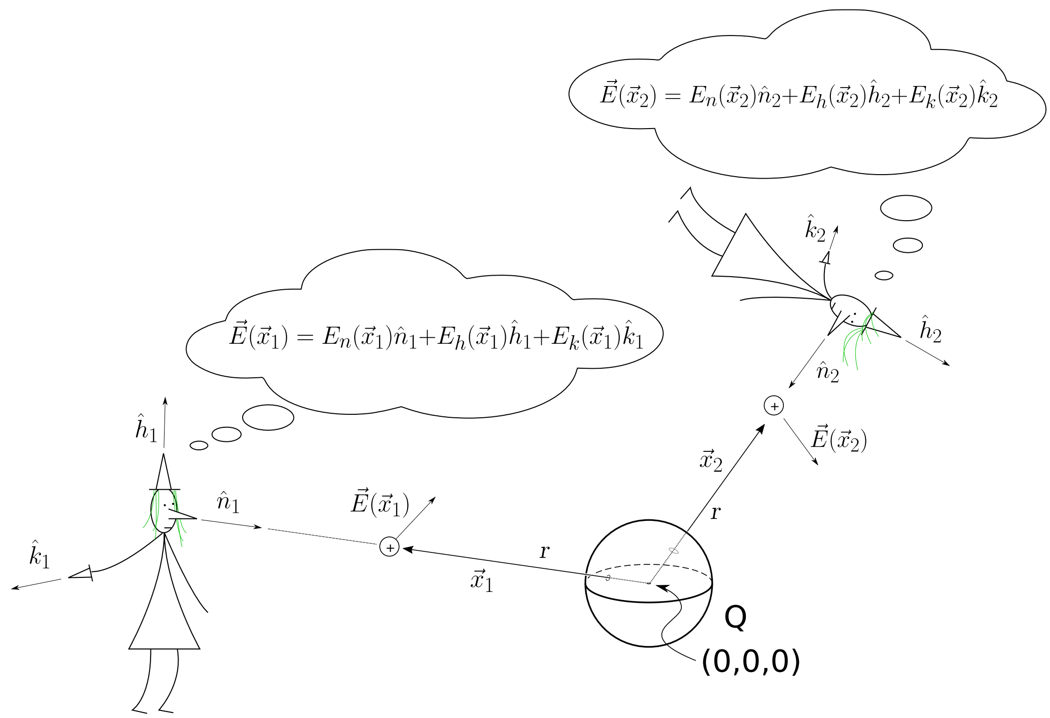

Figure 6: Electro-witches measuring the electric field at two points $\vec{x}_1$ and $\vec{x}_2$, both a distance r away from the sphere center.

distribution that their measurements to differ so that possibly

$$(E_n(\vec{x_1}), E_h(\vec{x_1}), E_k(\vec{x_1})) \neq (E_n(\vec{x_2}), E_h(\vec{x_2}), E_k(\vec{x_2}))$$

Note, however, the physical scenario each witch encounters in the spherical case. (It is helpful here to imagine yourself as one of these witches.) Both witches see a proton located directly in front of their nose and some distance r behind this a proton they see a sphere.

In light of this observation, we are forced to conclude their measurements of the proton acceleration (and thus the electric) **must be equal**, i.e.

$$(E_n(\vec{x_1}), E_h(\vec{x_1}), E_k(\vec{x_1})) = (E_n(\vec{x_2}), E_h(\vec{x_2}), E_k(\vec{x_2})) \tag{2}$$

This is the essence of symmetry arguments in physics. If two experimenters encounter an identical physical scenario we require that they obtain identical results.

Note, crucially, that equation (2) does **not** assert that the electric field *vectors* at the two different points are equal, i.e. we can not conclude $\vec{E}(\vec{x}_1) = \vec{E}(\vec{x}_2)$. Indeed we shall see that the vectors are not equal. We only assert that the *components* of the vector $\vec{E}(\vec{x}_1)$ as expressed in the first witch's internal coordinate system are the same as the components of the vector $\vec{E}(\vec{x}_2)$ expressed in the second witch's coordinate system. This fact, however, is enough to establish the equality the *lengths* of the two vectors (check this!). From here we note that our selection of $\vec{x}_1$ and $\vec{x}_2$ was *arbitrary*, only requiring $|\vec{x}_1| = |\vec{x}_2| = r$. Therefore, we can conclude that the *strength* $|\vec{E}(\vec{x})| \equiv E(\vec{x})$ of the electric field vector at any point $\vec{x}$ depends only on the length r of the vector $\vec{x}$, so that in the function $E(\vec{x})$ we can replace the argument $\vec{x}$ with just its length r, i.e. $E(\vec{x}) \to E(r)$. This completes step 1.(a) of the general procedure for determining the electric field of our charge distribution.

To execute step 1.(b) of the general procedure we consider a different scenario. Refer to figure 7. Instead of placing two witches in different locations, we:

1. have a witch measure the electric field at some location, which she reports in terms of the components E_n, E_h, E_k referenced to her internal coordinate system $\hat{n}, \hat{h}, \hat{k}$.
2. Then, we rotate the witch by 180° about her nose ($\hat{n}$) and have her repeat the measurement. This time she reports a result E'_n, E'_h, E'_k referenced to her new internal coordinate system $\hat{n}', \hat{h}', \hat{k}'$.

We now make three observations concerning these measurements:

1. As a result of the 180° rotation we find that
   (a) the vector $\hat{n}$ is unchanged so that $\hat{n}' = \hat{n}$, and
   (b) the vectors $\hat{h}$ and $\hat{k}$ are both *flipped* so that $\hat{h}' = -\hat{h}$ and $\hat{k}' = -\hat{k}$.
2. The 180° rotation was performed on the witch and *not* the electric field vector that she is measuring. Therefore, we assert that $\vec{E}' = \vec{E}$, or in other words

$$E_n\hat{n} + E_h\hat{h} + E_k\hat{k} = E'_n\hat{n}' + E'_h\hat{h}' + E'_k\hat{k}'$$

3. Finally, the physical scenario that the witch encounters after being flipped upside down is identical to one she encountered before she was flipped. Therefore, we also assert that the results of her measurements be identical so that

$$E'_n = E_n,\ E'_h = E_h,\ E'_k = E_k$$

These three observations can be combined to arrive at the following conclusions:

$$E_n = E_n,\ E_h = -E_h,\ E_k = -E_k$$

The first of these equations tells us nothing new (any number is obviously equal to itself), but the following two tell us that the E_h and E_k components are equal to their negatives. Well the only number equal to its negative is zero, so we can conclude

$$E_h = E_k = 0$$

Or, in other words, the electric field at any point $\vec{x}$ in space points parallel to $\hat{n}$. Since we arranged in the beginning of the problem to always have the witch's nose pointing parallel to $\vec{x}$, this also means

$$\vec{E}(\vec{x}) \parallel \vec{x}$$

so that we find the electric field points at all points in the direction away from the sphere's center[2].

This completes step 1.(b) of our general procedure. We finish the derivation with step 2 of our general procedure, which is illustrated in figure 8. We have constructed a hypothetical (or "Gaussian") surface in the shape of a sphere using N small patches $\Delta\vec{A}_i\, i = 1, 2, \ldots, N$, each located at a position $\vec{x}_i$. The flux Φ through the entire surface is given by

$$\Phi = \sum_{i=1}^{N} \vec{E}(\vec{x}_i) \cdot \Delta\vec{A}_i$$

[2] Actually it may in some cases point in the *opposite* direction, i.e. *towards* the sphere's center, depending on the charge of the sphere.

Since the patches $\Delta\vec{A}(\vec{x})_i$ are all tangent to the sphere, their surface normal, which is defined to be the *direction* of the $\Delta\vec{A}$, points away from the center of the sphere, i.e.

$$\Delta\vec{A} \parallel \vec{x}_i$$

Recall from step 1.(b) in the general procedure where we found that the electric field also points away from the sphere's center, so that for any patch center $\vec{x}_i$ we have

$$\vec{E}(\vec{x}_i) \parallel \vec{x}_i$$

so that

$$\vec{E}(\vec{x}_i) \parallel \Delta\vec{A}$$

The dot product $\vec{a} \cdot \vec{b}$ of two parallel vectors is simply the product of their magnitudes $|\vec{a}||\vec{b}|$ so we have

$$\vec{E}(\vec{x}_i) \cdot \Delta\vec{A}_i = |\vec{E}(\vec{x}_i)||\Delta\vec{A}_i| = E(r_i)\Delta A_i$$

where r_i is the length $|\vec{x}_i|$ of the vector $\vec{x}_i$ and $\Delta A_i \equiv |\Delta\vec{A}_i|$ is again just the area of the patch.

The patch centers are all equidistant from the sphere center so that r_i is independent of i. Setting then $r_i \to r$ for every i we have

$$\begin{aligned}\Phi =& \sum_{i=1}^{N} \vec{E}(\vec{x}_i) \cdot \Delta\vec{A}_i \\ =& \sum_{i=1}^{n} E(r)\Delta A_i \\ =& E(r) \sum_{i=1}^{n} \Delta A_i \\ =& E(r)4\pi r^2\end{aligned}$$

where in the last step we notice that the sum of the areas of all the patches should be equal to the surface area $4\pi r^2$ of a sphere of radius r.

From Gauss' law we can say the flux Φ is also equal to $\frac{1}{\epsilon_o}$ times the charge enclosed by the surface, which is simply Q, the charge of the charged sphere. Therefore we find

$$E(r)4\pi r^2 = \frac{Q}{\epsilon_o}$$

or

$$E(r) = \frac{Q}{4\pi\epsilon_o r^2}$$

Combining this result with our knowledge that the electric field $\vec{E}(\vec{x})$ lies in the direction of the unit vector $\hat{x} \equiv \frac{\vec{x}}{r}$ (remember $r \equiv |\vec{x}|$) we find

$$\vec{E}(\vec{x}) = \frac{Q}{4\pi\epsilon_o r^2}\hat{x}$$

So we find that the electric field of a sphere with charge Q is equal to electric field of a particle with the same charge Q positioned at the sphere's center. The (somewhat unsurprising) result may familiar from the study of gravity, where an object orbiting under a spherical object of mass M traced the same trajectory as an object orbiting a point particle with the same mass M.

Careful though! In the execution of step 2 we made the (implicit) assumption that our Gaussian sphere *enclosed* the charged sphere. How does our analysis change if our charged sphere is a thin *shell* of, say, radius b, and we construct a Gaussian sphere of radius $a < b$, so that the Gaussian sphere is *inside* the charged shell? The analysis in step 1. of our general procedure still applies, but what modification must we make in step 2.? What if our charge distribution consists of two thin shells of radius a and b containing charge Q_a and Q_b, respectively, and we construct a Gaussian sphere of radius c where $a < c < b$?

3.2 Cylindrical distributions

Next we consider charged cylinders that are infinitely long. The charge of the cylinder is characterized by a *linear charge density* λ so that the charge Q contained in some section of the cylinder of length l is $Q = \lambda l$. We further assume that the linear charge density λ is *constant* along the length of the cylinder.

The strategy for determined the electric field generated by this charge distribution will be nearly identical to the case of the sphere; we only have to make a couple adjustments to take account differences in their symmetry.

Refer to figure 9. We adopt the conventions that any witch's hat $\hat{h}$ always point parallel to the cylinder's axis of symmetry, and that her nose lies along the line joining the hypothetical proton to the cylinder axis. In the illustrated scenario one witch measures the electric field vector at a location some distance r from the cylinder and another witch measures the field vector at another position obtained by

- a rotation Δ *about* the cylinder's axis of symmetry (pointing up and down in the figure), followed by
- a displacement Δz *along* the symmetry axis.

Note that this implies the second point also lies a distance r from the cylinder axis. Because of the cylinder's symmetry, we find, like we did in part 1.(a) of the spherical case, that the witches encounter identical physical scenarios and thus measure the same components for the electric field in their internal coordinate systems. This in turn implies that the *strength* $|\vec{E}|$ of the electric field vector is the same at either point. We conclude then that the strength of the electric field generated by a cylindrical charge distribution at some point $\vec{x}$ depends only on the distance r that point is from the cylinder axis, so that

$$E(\vec{x}) \to E(r)$$

This completes step 1.(a) of the general procedure.

The argument for step 1.(b) for the cylindrical case is identical to the spherical case. Refer to figure 10. The witch measures the electric field at some point, and measures it again after being rotated about her nose ($\hat{n}$) by 180°. The physical scenario she encounters is identical[3] in either orientation, so we can reuse the argument detailed in the spherical case to conclude

$$E_h = E_k = 0$$

so that the electric field points at any point directly *away* from the cylindrical axis[4]. This completes step 1.(b) of our general procedure.

Step 2 of the general procedure also works out similar to the spherical case. Refer to figure 11. We have drawn in blue a cylindrical Gaussian surface of length l and radius r. The patch vectors $\Delta\vec{A}$ composing the curved wall of the have their normal vectors pointing *away* from the cylinder axis and thus parallel to

[3]Note that this would **not** be the case if there were, for instance, some electrical *current* flowing down the cylinder. In this scenario the witch would observe a *down*wards flowing current in her first orientation and an *up*wards flowing current after being flipped upside-down.

[4]or directly *towards* it, depending on whether the charge density λ is positive or negative.

the electric field. The total flux Φ through the curved wall is thus equal to the magnitude of the electric field $E(r)$ at the wall multiplied by the surface area of the curved wall, which is $2\pi rl$, so that

$$\Phi = E(r) \times 2\pi rl$$

The situation is different for the patch vectors composing the flat caps on the top and bottom of the Gaussian surface. These patches point *along* the cylinder axis, which is perpendicular to the direction of the electric field. Since the dot product of perpendicular vectors is always zero, we find that the end caps do not contribute to the electric flux. Therefore we find that

$$\Phi = 2\pi rlE(r)$$

Since the cylinder encloses a length l of the charged cylinder, and the charge contains a quantity λ charge per unit length, we obtain from Gauss' law

$$\Phi = \lambda l/\epsilon_o$$

Equating the alternative expressions for the flux we arrive at

$$2\pi rlE(r) = \lambda l/\epsilon_o$$

or

$$E(r) = \frac{\lambda}{2\pi r\epsilon_o}$$

If for a point $\vec{x}$ we let $\hat{r}$ be the unit vector pointing directly away from the cylinder axis we arrive at the complete expression for the electric field $\vec{E}(\vec{x})$ produced at a point $\vec{x}$ by an infinitely long cylindrical of linear charge density λ:

$$\vec{E}(\vec{x}) = \frac{\lambda}{2\pi r\epsilon_o}\hat{r} \tag{3}$$

where again r is the distance from the point $\vec{x}$ to the cylinder axis and $\hat{r}$ is a unit vector pointing directly away from the axis. We could alternatively express this by defining for each point $\vec{x}$ the vector $\vec{r}$ to be the shortest vector we can draw from the cylinder axis to $\vec{x}$, i.e. the vector that leaves the cylinder axis at a right angle and ends at the point $\vec{x}$. Then we would say $r \equiv |\vec{r}|$ and $\hat{r} \equiv \vec{r}/r$. See figure 12 for an illustration. Again, we want to be careful applying the conclusion in equation (3). If, for instance, we had a thin cylindrical tube of radius b for our charge distribution, and we constructed a Gaussian surface of radius $a < b$, what would we conclude about the electric field inside the tube?

3.3 Planar distribution

Finally we consider a planar distribution of charge. The plane extends infinitely far in both lateral directions[5]. The charge on the plane is characterized by its *surface charge density* σ so that the charge over any region of the surface is given by σA, where A is the area of the region.

As with the previous two cases we begin by analyzing the electric field measured at two different points. Refer to figure 13. We adopt the convention for the planar distribution that the witch's nose always lie colinear with the shortest line joining the plane to the point $\vec{x}$ where the witch measures the electric field. When we additionally take into account the convention established earlier requiring the witch's nose to point towards $\vec{x}$, we find that her nose $\hat{n}$ points always normal to the plane.

With these conventions established we then find that we can displace the observation point $\vec{x}$ by any amount Δx and Δy in the lateral directions without modifying the physical scenario observed by the witch.

[5] We can just as well consider the plane to be a circle of infinite radius.

This tells us that the strength of the electric field $E(\vec{x})$ can depend only on the distance d from the point $\vec{x}$ to the plane, i.e. $E(\vec{x}) \to E(d)$. This completes step 1.(a) of the general procedure.

Step 1.(b) of the general procedure is carried out in exactly the same way as the cylindrical and spherical distributions. Since we can rotate a witch by 180° about her nose $\hat{n}$ without modified the physical scenario she observes, we can conclude

$$E_h = E_k = 0$$

so that the electric field vector at any point is directed along the plane's normal.

For step 2.(b) we construct a cylindrical Gaussian surface of length $2d$ and endcaps of area A. The Gaussian cylinder intersects the charged plane halfway along the cylinder's length. See figure 14 for an illustration.

To compute the electric flux Φ through the Gaussian cylinder we observe that, in constract with the charged cylinder, it is now the surface patch vectors comprising the end caps which point parallel to the electric field, yielding a flux

$$\Phi = 2AE(d)$$

The patch vectors comprising the curved walls do not contribute to the electric flux since they now point perpendicular to the electric field.

The charge enclosed by the Gaussian cylinder is the cylinder's cross-sectional area A multiplied by the surface charge density σ. From Gauss' law we can then conclude

$$2AE(d) = \sigma A/\epsilon_o$$

or

$$E = \frac{\sigma}{2\epsilon_o}$$

so that the electric field strength $E(\vec{x})$ turns out to be independent of not only the lateral coordinates of $\vec{x}$, but also the separation d of the point $\vec{x}$ from the surface. Leaving behind our electro-witches, we now let $\hat{n}$ stand for the unit vector normal to the charged plane so that

$$\vec{E} = \frac{\sigma}{2\epsilon_o}\hat{n}$$

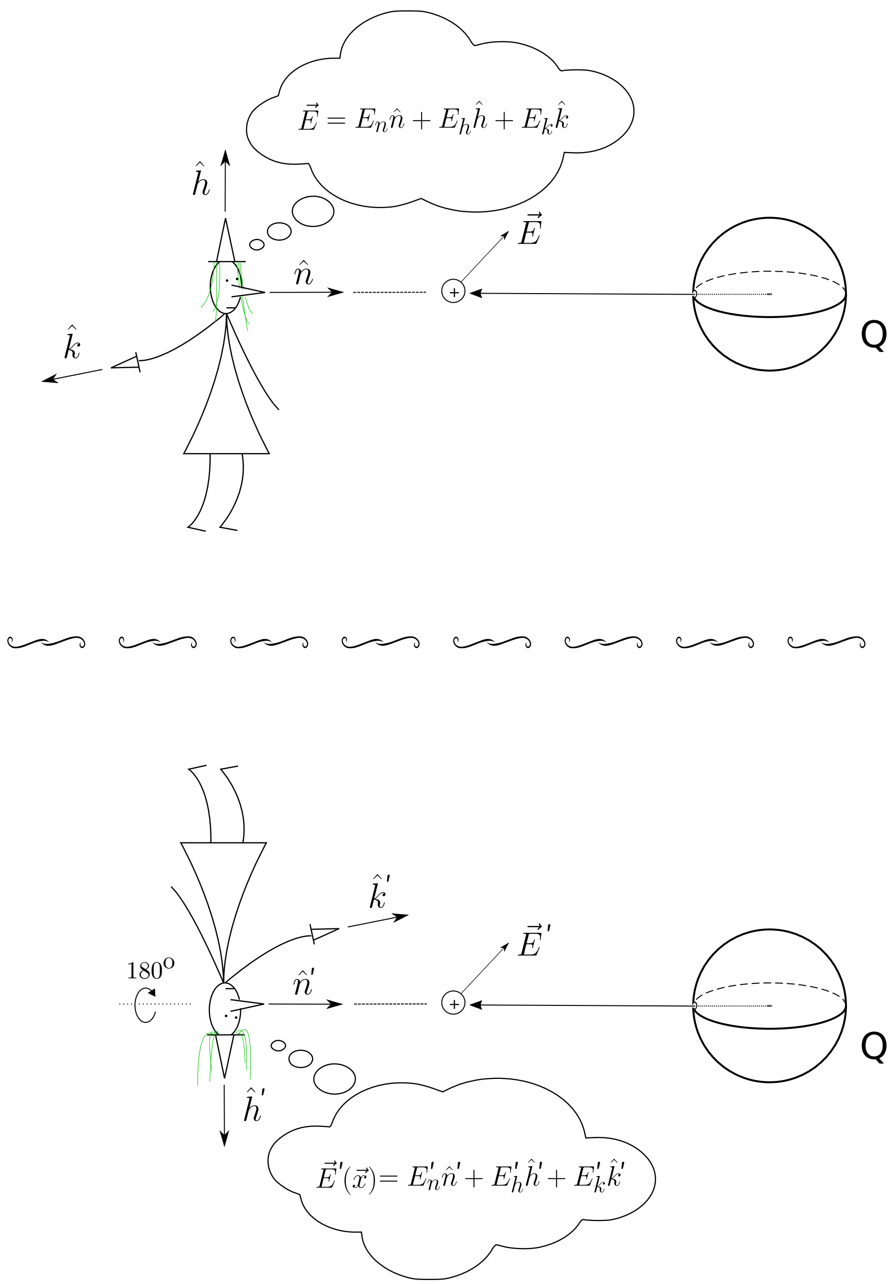

Figure 7: A single electro-witch measuring the electric when she is right-side up (top half) and up-side down (bottom half).

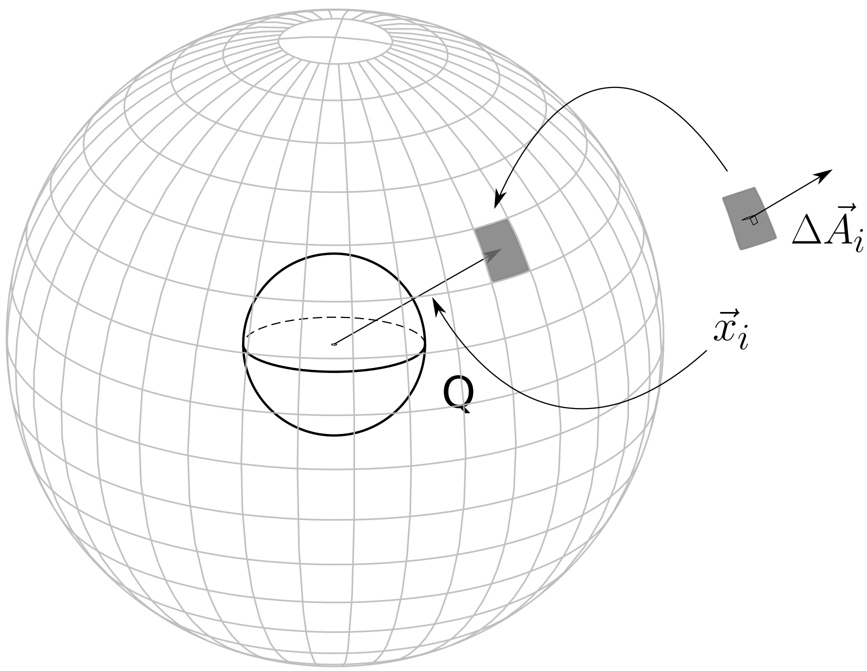

Figure 8: Charged sphere surrounding by spherical Gaussian surface. The Gaussian sphere is approximated by many flat patch vectors $\Delta\vec{A}_i(\vec{x}_i)$, $i = 1, 2, \ldots, N$.

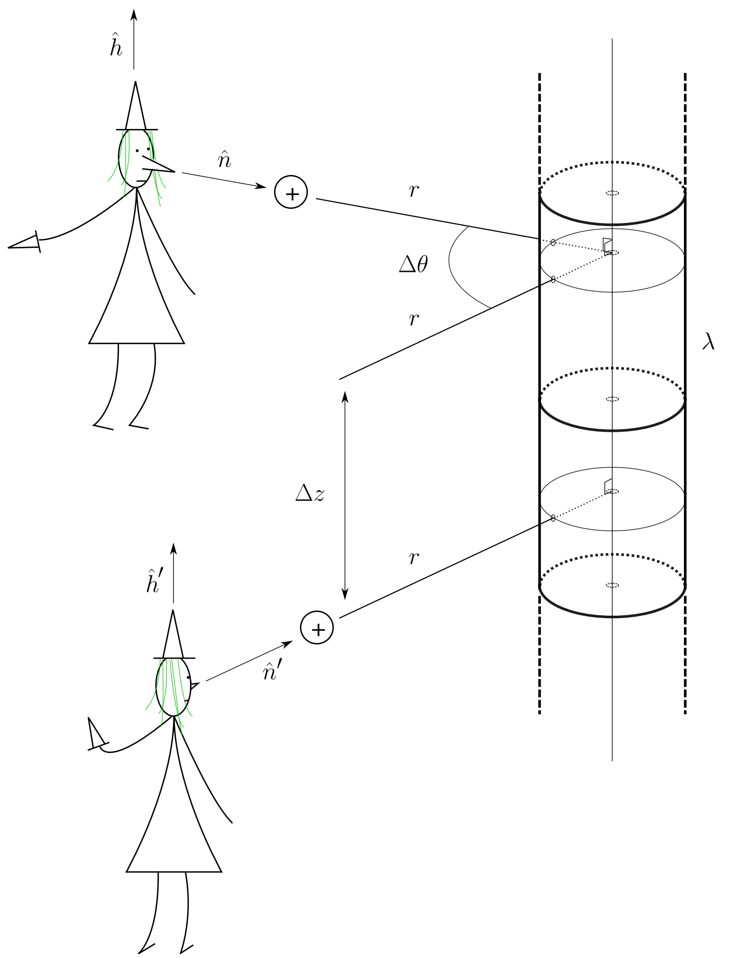

Figure 9: Two witches measure the electric field produced at two points by an infinitely long cylindrical charge distribution with linear charge density λ.

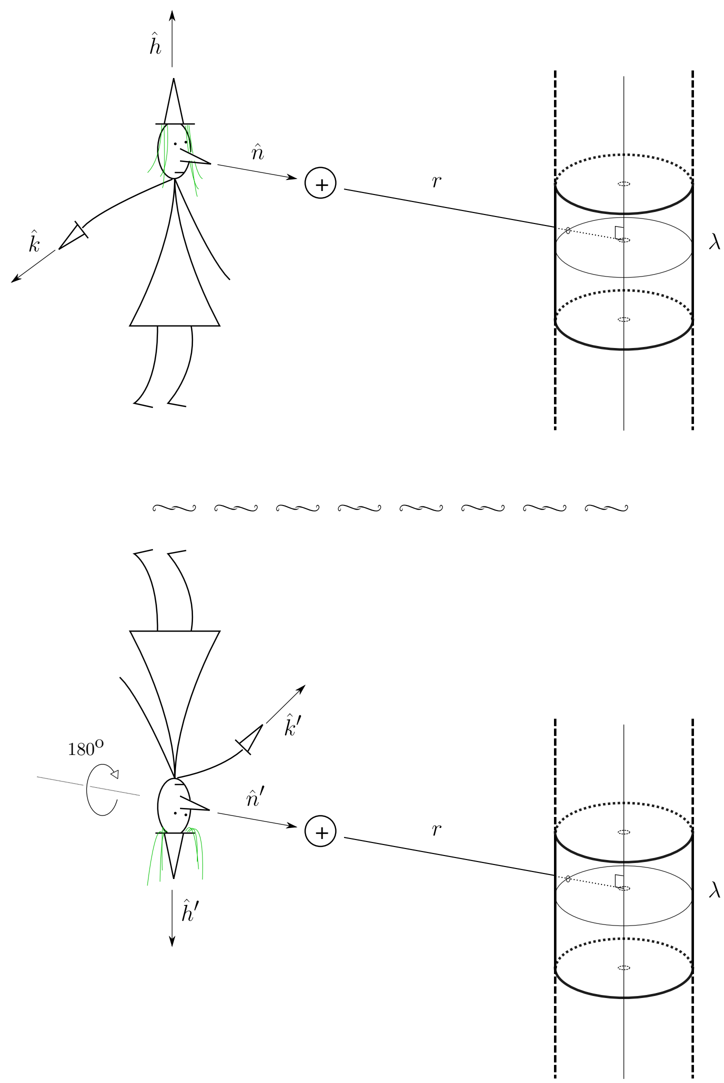

Figure 10: A witch measures the electric field generated by a cylindrical charge distribution in two different orientations: right-side up (upper half) and up-side down (lower half).

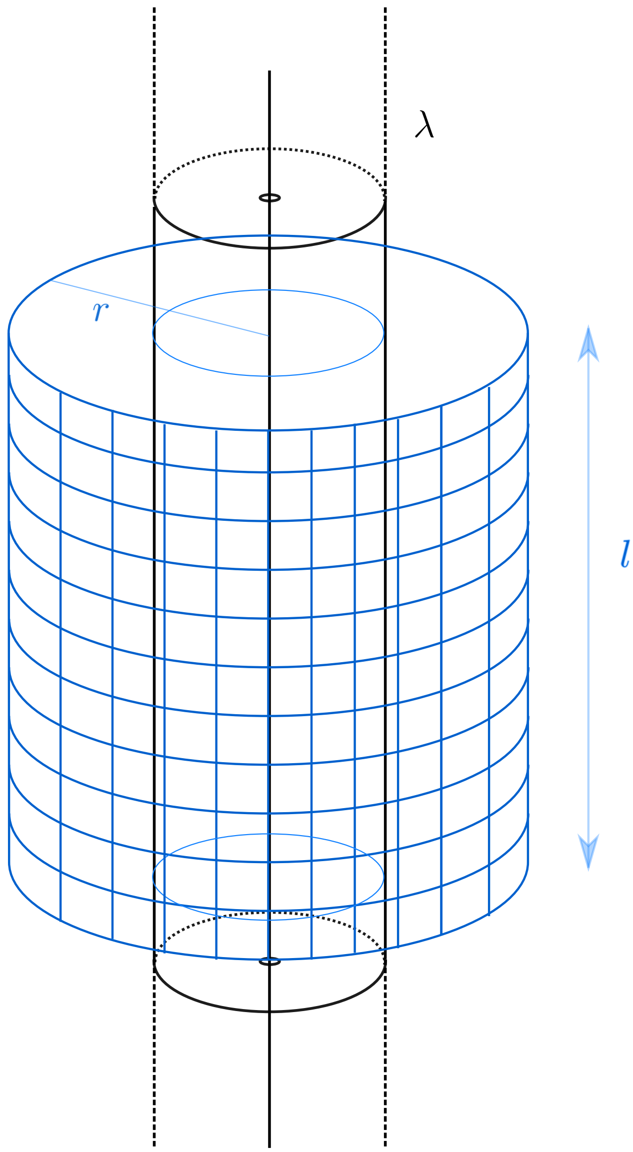

Figure 11: A cylindrical Gaussian surface of length l and radius r drawn around an infinitely long cylindrical charge distribution

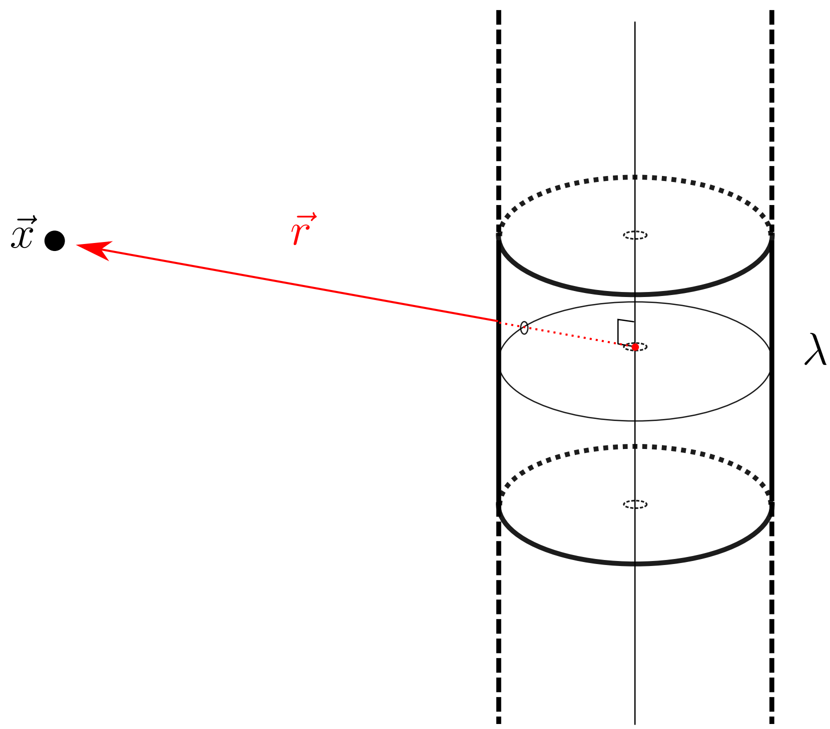

Figure 12: Illustration of the vector $\vec{r}$ associated with a point $\vec{x}$

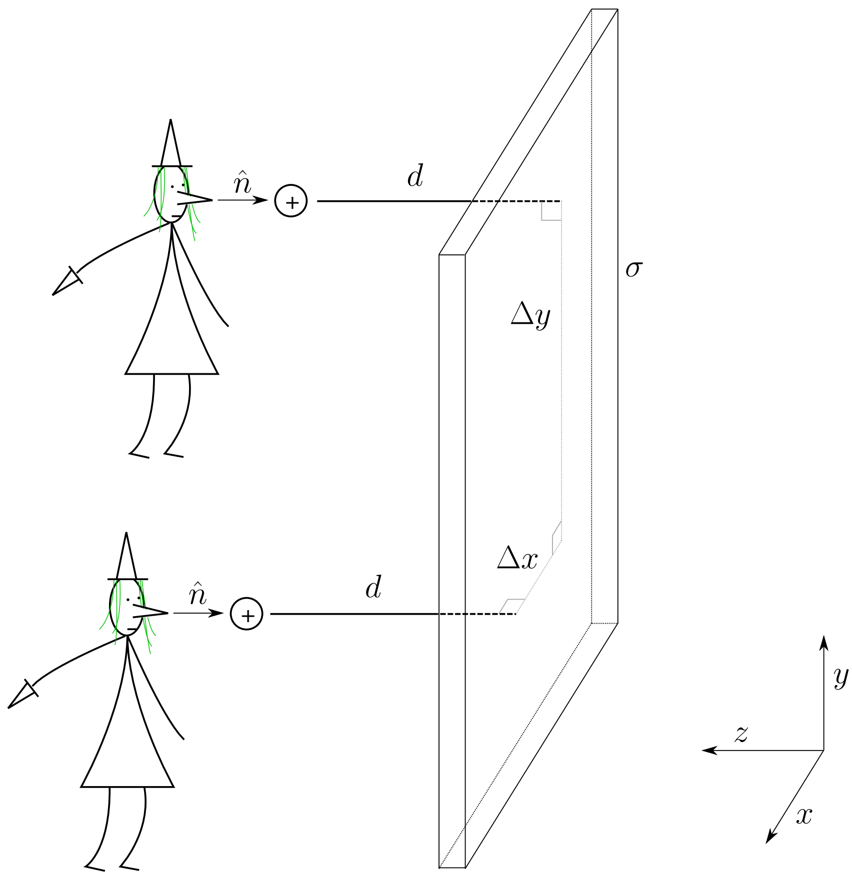

Figure 13: Witches measuring the electric field generated by an infinite charged plane.

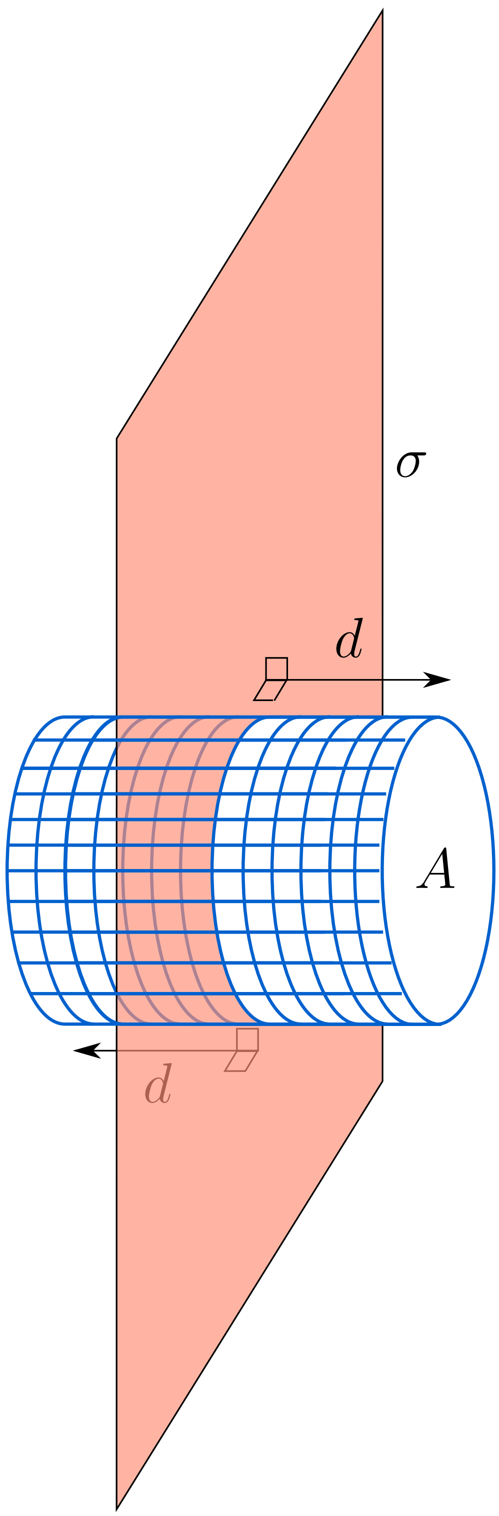

Figure 14: Gaussian surface (in blue) used to compute the electric field generated by the charged plane (in red).