302L S20 Essentials

1 Electric charge

- *charge conservation*: Electric charge is neither created nor destroyed. If the net charge contained in some region at some initial time is different at some later time, we conclude that the missing charge left the system.
- *charge quantization*: Charge comes in integer (positive or negative) multiples of some fundamental unit called e, which in SI units has a value

$$e = 1.6022 \cdot 10^{-19} \text{ C}$$

  Ordinary matter is made out of electrons, protons, and neutrons.

  - The electron has a charge of $-e$.
  - The proton has a charge of $+e$.
  - The neutron has a charge of zero.
- Materials fall into one of two categories based on the nature of the charge carriers in the material:
  - *Conductors* have charge carriers that move freely about the material. e.g. metals, electrolytic solutions (saltwater), plasmas.
  - *Insulators* have charge carriers that are only able to displace themselves a very short distance in response to electrical forces. e.g. glass, plastic, pure water, air.
- Objects can become charged by:
  - friction (triboelectricity)
  - conduction
  - induction

  see notes for details.

2 Coulomb's Law

- *Coulomb's law*: The force $\vec{F}$ exerted *by* a particle of charge q' located at a position $\vec{x}'$ *on* a particle of charge q located at a position $\vec{x}$ is given by:

$$\vec{F} = \frac{qq'}{4\pi\epsilon_o|\vec{x}-\vec{x}'|^2} \cdot \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|}$$

where

$$\epsilon_o = 8.85 \cdot 10^{-12}\ \mathrm{C^2 m^{-2} N^{-1}}$$

is a constant known as the *permittivity of free space*. We sometimes make the substitution $k \equiv \frac{1}{4\pi\epsilon_o} \approx 9 \cdot 10^9\ \mathrm{Nm^2C^{-2}}$ in which case Coulomb's law reads:

$$\vec{F} = \frac{kqq'}{|\vec{x} - \vec{x}'|^2} \cdot \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}$$

More simply we could state:

> *The force F exerted by a particle of charge q' on a particle of charge q is $\frac{kqq'}{r^2}$, where r is the distance between the particles. The direction of the force is parallel to the line joining the particles, and is repulsive or attractive depending on whether the particles have similar or opposite charge.*

- *Superposition principle*: The force exerted by a collection of charged particles $B, C, D, \ldots$ on a charged particle A is equal to the (vector!) sum of the individual forces applied by each particle in the collection.

3 Electric Field

- We can associate with any collection of charged particles $B, C, D, \ldots$ an *electric field* $\vec{E}(\vec{x})$. The electric field is a *function* taking in a location in space $\vec{x}$ and returning a *vector* $\vec{E}(\vec{x})$, called the *electric field vector at* $\vec{x}$. The electric field is defined so that, if we were to place a particle A of charge q at a location $\vec{x}$, the collection of charged particles $B, C, D, \ldots$ would exert a force $\vec{F}$ on the particle A given by

$$\vec{F} = q\vec{E}(\vec{x})$$

  We say that the collection $B, C, D, \ldots$ *generates* the electric field $\vec{E}(\vec{x})$.

- The electric field $\vec{E}(\vec{x})$ generated by a point charge of charge q' located at a point $\vec{x}'$ is given by:

$$\vec{E}(\vec{x}) = \frac{q'}{4\pi\epsilon_o|\vec{x} - \vec{x}'|^2} \cdot \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|}$$

- The superposition principle holds for electric fields, i.e. to obtain the electric field generated by multiple charged particles we add up the electric fields generated by each particle individually.

4 Gauss' Law

- *Electric flux*: Given an electric field $\vec{E}(\vec{x})$ and some hypothetical ("Gaussian") surface, we can define a number Φ known as the *electric flux* of the electric field through the surface by the following procedure:

  - Divide up the surface into regions $R_1, R_2, \ldots, R_N$ so that the quantity

$$\vec{E}(\vec{x}) \cdot \hat{n}(\vec{x}) \equiv \Phi'_i \tag{1}$$

    is the *same* for every point $\vec{x}$ on a particular region R_i of the surface. We can refer to Φ'_i as the "electric flux density" of the region R_i. The vector $\hat{n}(\vec{x})$ is the unit vector pointing *perpendicular* to the surface at the point $\vec{x}$.

– The electric flux Φ through the entire surface is then given by

$$\Phi = \sum_{i=1}^{N} \Phi'_i A_i \tag{2}$$

where A_i is the *area* of the region R_i.

This definition is abstract but in this class it will always be (somewhat) obvious how to select the regions $R_1, R_2, \ldots, R_N$.

For example, when solving for the flux through a box due to a *constant* electric field, we can choose as our regions $R_1, R_2, \ldots, R_6$ the six faces of the box.

Similarly, for spherical charge distributions and a concentric spherical Gaussian surface, we find, because of symmetry, the flux density Φ' is constant over the entire surface. We do not then need to divide our Gaussian surface at all, i.e. to apply equation (2) we set $N = 1$ and we have a single region R_1 equal to the entire sphere with its associated area A_1 and flux density Φ'_1.

For cylindrically symmetric charge distributions we use a cylindrical Gaussian surface. This surface divides into three regions of constant flux density. Do you recall what these regions are? Which regions have non-zero flux density?

- *Gauss' Law*: The electric flux Φ through a *closed* surface is related to the charge q_{enc} *enclosed* by the surface by the following equation:

$$\Phi = q_{enc}/\epsilon_o$$

Note that this assumes the convention that every normal vector $\hat{n}(\vec{x})$ on the Gaussian surface points *away* from the volume the surface encloses.

What is remarkable about Gauss' Law is that the electric field $\vec{E}(\vec{x})$ used to compute the flux (equation (1)) is generated by *all* the charges in the system, even the ones outside the Gaussian surface.

5 Applications of Gauss' Law

1. *Sphere*: A spherical distribution of charge generates an electric field $\vec{E}(\vec{x})$ given by

$$\vec{E}(\vec{x}) = \frac{Q(r)}{4\pi\epsilon_o r^2}\hat{x}$$

where:

- We assume the center of the spherical distribution is at the origin $(0, 0, 0) \equiv \vec{0}$
- $r \equiv |\vec{x}|$
- Q(r) is the amount of charge that lies within r of the sphere center.
- $\hat{x} \equiv \frac{\vec{x}}{r}$ is a unit vector pointing away from the sphere center.

Note that $Q(r)$ is not necessarily the entire charge of the distribution, but depends on *where* you are measuring the electric field, i.e. depends on $\vec{x}$. Recall the hollow shell example from class.

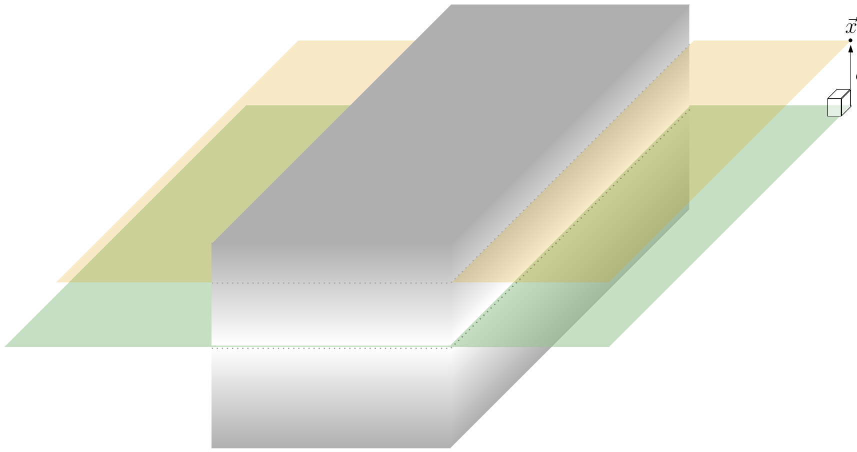

Figure 1: The infinite plate distribution. The green plane indicates the middle of the distribution, and the orange plane are all the points lying a distance d above the middle.

2. *Cylinder*: An infinitely long cylindrical distribution of charge generates an electric field $\vec{E}(\vec{x})$ given by:

$$\vec{E}(\vec{x}) = \frac{\lambda(r)}{2\pi\epsilon_o r}\hat{r}$$

   where:

   - $\vec{r}$ is the shortest vector joining the cylinder axis to $\vec{x}$. Alternatively, it is the vector leaving the cylinder axis at a right angle and ending at the point $\vec{x}$.
   - $r \equiv |\vec{r}|$ is distance from the cylindrical axis to $\vec{x}$.
   - $\lambda(r)$ is the charge per unit length lying within a distance r of the cylinder axis.
   - $\hat{r} \equiv \frac{\vec{r}}{r}$ is a unit vector pointing away from the cylindrical axis.

   We note that, similar to the spherical case, $\lambda(r)$ is not necessarily the entire linear charge density of the cylindrical charge distribution, but depends on your distance from the cylindrical axis, i.e. depends on $\vec{x}$.

3. *Plane*: A symmetric plate (flat) distribution of charge that extends infinitely far in both directions generates an electric field $\vec{E}(\vec{x})$ given by:

$$\vec{E}(\vec{x}) = \frac{\sigma(d)}{2\epsilon_o}\hat{d}$$

   where (see figure 1:

   - $\vec{d}$ is the shortest vector joining the middle of the plate to $\vec{x}$. It is also the vector leaving the middle of the plate in the direction normal to the plate and ending at $\vec{x}$.
   - $d \equiv |\vec{d}|$ is the distance from the middle of the plate.
   - $\sigma(d)$ is the charge per unit area lying within a distance d above or below the middle of the plate.
   - $\hat{d} \equiv \frac{\vec{d}}{d}$ is a unit vector pointing away from the plate.

Again note that not necessarily all the surface charge density is located within some distance d of the middle of the plate so the strength of the electric field depends in general on where you are, i.e. depends on $\vec{x}$. If this point is not clear, think about the electric field *in between* two sheets of equal surface charge density, spaced apart by some distance. Check that the above equation predicts, as we expect, an electric field of zero between the two sheets.

In the case where the plate is *infinitely thin*, then for any $\vec{x}$ the associated $\sigma(d)$ is the full surface charge density $\equiv \sigma$ of the plate so that:

$$\vec{E}(\vec{x}) = \frac{\sigma}{2\epsilon_o}\hat{d}$$

6 Conductors and the Electric Field

The following applies to electrical conductors *at equilibrium* (i.e. to electro*statics*).

1. The electric field *inside* a conductor is zero.
2. The charge density *inside* a conductor is also zero.
3. The electric field $\vec{E}(\vec{x})$ at a point $\vec{x}$ at the surface of a conductor is given by[1]

$$\vec{E}(\vec{x}) = \frac{\sigma(\vec{x})}{\epsilon_o}\hat{n}(\vec{x})$$

   - $\sigma(\vec{x})$ is the surface charge density at the point $\vec{x}$.
   - $\hat{n}(\vec{x})$ is the unit vector pointing normal to the surface at $\vec{x}$.

4. Surface charge tends to concentrate at point of high curvature, e.g. pointy tips and sharp edges. Therefore the electric field is strong at (and immediately outside) high curvature regions of the surface of a conductor.
5. For most conductors, strong chemical forces keep charged particles from exiting the surface. We discussed three ways of overcoming these forces in the case of electrons at the surface of a metal conductor:
   - *thermionic emission*: Emission of electrons at high temperatures.
   - *photoemission*: Emission of electrons induced by absorption of short wavelength light (photons).
   - *field emission*: Emission of electrons induced by (very) high surface charge density.

   See the lecture notes for more details.

7 Work and Potential Energy

- The *work* (denoted W) done on a particle on some path is a quantity that depends on the path and the forces $\vec{F}(\vec{x})$ the particle feels at the points $\vec{x}$ on the path.

[1] This expression for the electric field also holds for points immediately above $\vec{x}$. In other words for any point $\vec{x}$ on the surface of a conductor we also have

$$\vec{E}(\vec{x} + \Delta x\hat{n}) \approx \vec{E}(\vec{x})$$

where Δx is some small distance.

- *Work-energy principle*: The work W done along a particle's *trajectory* is equal to the change ΔKE in a particle's kinetic energy:

$$KE' = KE_o + W$$

where KE_o and KE' are the particle's initial and final kinetic energies, respectively.

- For a path consisting of a straight line segment from a point $\vec{x}_A$ to another point $\vec{x}_B$ where at all points along the line segment the force is a constant $\vec{F}_o$, the work W done is

$$W = \vec{F}_o \cdot (\vec{x}_B - \vec{x}_A)$$

- For certain force fields, including the Coulomb force field generated by charged particle, the work done along a path depends only on the path's endpoints, and not on the details of the path in between. In this case we can define a function $U(\vec{x})$ called the *potential energy* so that the work $W_{A \to B}$ going along a path from a point $\vec{x}_A \equiv A$ to a point $\vec{x}_B \equiv B$ is given by

$$W_{A \to B} = U(\vec{x}_A) - U(\vec{x}_B) \equiv U_A - U_B$$

- *Energy conservation*: By combining the work-energy principle and the concept of the potential energy, we find that the sum $KE + U$ of a particle's kinetic and potential energy is *fixed* for all times. This means that:
  - if a particle has an intial kinetic energy KE_o and initial potential energy U_o, and
  - a final kinetic energy KE' and final potential energy U', then
  - $KE_o + U_o = KE' + U'$

8 Electric Potential

- The electric potential $V(\vec{x})$, like the electric field $\vec{E}(\vec{x})$, is a way of describing the influence of a charge distribution on other charged particles.

- The potential energy $U(\vec{x})$ of a particle with charge q interacting with a charge distribution generating an electric potential $V(\vec{x})$ is given by $U(\vec{x}) = qV(\vec{x})$

- Electric potential has units of volts (V = 1J/C) and is alternatively referred to as a *voltage*.

- The electric potential difference $V_2 - V_1$ between two points $\vec{x}_1$ and $\vec{x}_2$ in a region of constant electric field $\vec{E}_o$ is given by

$$V_2 - V_1 = -\vec{E}_o \cdot (\vec{x}_2 - \vec{x}_1) = -E_o d \cos\theta$$

where
  - $E_o = |\vec{E}_o|$ is the strength of the field,
  - $d = |\vec{x}_2 - \vec{x}_1|$ is the length of the line segment $\vec{x}_1 \to \vec{x}_2$, and
  - θ is the angle made between the field vector and the line segment.

- Positive charge seeks decreasing electric potential, while negative charge seeks increasing electric potential.

- The electric potential generated by a point particle of charge q' is given by

$$V(r) = \frac{q'}{4\pi\epsilon_o r}$$

where r is the distance away from the point charge.

- Expressions exist for the electric potential generated by some other common charge distributions (sheets, spherical shells, cylindrical shells).
- Electric potential obeys a superposition principle.

9 Capacitance

- In equilibrium, the electric potential is the same everywhere on a conductor.
- A capacitor is a pair of conductors.
- The capacitance C of a conductor is the ratio Q/V where V is the potential difference between the conductors when there is a charge Q on one conductor and an opposite charge $-Q$ on the other.
- In SI units capacitance is given in Coulombs per Volt, i.e. C/V. A capacitance of 1 C/V is known as a Farad, abbreviated F.
- The capacitance of a parallel plate capacitor is $\frac{\epsilon_o A}{d}$, where A is the area of the plates and d is the distance between them.
- Capacitors are used to store charge.

10 Capacitors in Parallel and Series

- Capacitors in parallel:
  - By combining two capacitors with capacitances C_A and C_B in parallel we obtain a new capacitor with capacitance $C_A + C_B$
  - The charges Q_A and Q_B on capacitors C_A and C_B joined in parallel are proportional to their capacitances; i.e. $Q_A/Q_B = C_A/C_B$.
- Capacitors in series:
  - By combining two capacitors with capacitances C_A and C_B in series we obtain a new capacitor with capacitance $\left(\frac{1}{C_A} + \frac{1}{C_B}\right)^{-1}$
  - The charges Q_A and Q_B on capacitors C_A and C_B joined in series are the same and equal to the total charge Q across the series combination.

11 Energy Stored on a Capacitor

- The energy E on a capacitor with capacitance C at a voltage V (and thus holding a charge $Q = CV$) is given by

$$\frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV$$

12 Vectors

Note: this section is meant for reference in case you are confused about certain details regarding vectors and their application to physics problems. You will only be tested on your physics knowledge and your ability to apply mathematics to solve physics problems. You will not, for instance, be asked to prove mathematical mathematical theorems.

A *vector* is a sequence of numbers. In this class we encounter three dimensional vectors, which are sequences of three numbers. We oftentimes will refer to a general vector $\vec{v}$, which represents an *arbitrary* sequence of three numbers, (v_x, v_y, v_z). We sometimes also use a boldface font $\mathbf{v}$ for the vector (v_x, v_y, v_z).

12.1 vector operations

We can *add* two vectors $\vec{v} = (v_x, v_y, v_z)$ and $\vec{w} = (w_x, w_y, w_z)$ to get another vector:

$$\vec{v} + \vec{w} = (v_x + w_x, v_y + w_y, v_z + w_z)$$

and we can *scale* a vector $\vec{v}$ by a number c to get another vector:

$$c\vec{v} = (cv_x, cv_y, cv_z)$$

We can also take the *dot product* of two vectors $\vec{v}$ and $\vec{w}$ to get a number:

$$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z$$

Finally, we can take the *length* of a vector $\vec{v}$ to get a number:

$$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$$

12.2 connection between vectors and physical space

Any point P in our three-dimensional physical world can be assigned a vector $\vec{p} = (p_x, p_y, p_z)$ by the following procedure (refer to figure 2):

1. Select some special point O in space, which we will call the *origin*.
2. Pick three mutually perpendicular lines, all three intersecting at the origin. Call the first one the x-direction, the second the y-direction, and the third the z-direction.
3. The vector $\vec{p} \equiv (p_x, p_y, p_z)$ associated with a point P in space is determined by the respective lengths of the three line segments on a three step path consisting of:
   (a) (red) A line segment from the origin moving in the x-direction, followed by
   (b) (blue) a line segment moving in the y-direction, followed by
   (c) (green) a line segment moving in the z-direction,

   with the additional rule that when you move *backwards* in some direction you multiply the corresponding line segment by -1.

In physics it does not matter where you pick the origin O or the three coordinate axes x, y, z so long as you are *consistent* with your choice. Note that according to these rules the origin O has the corresponding vector $\vec{0} = (0, 0, 0)$.

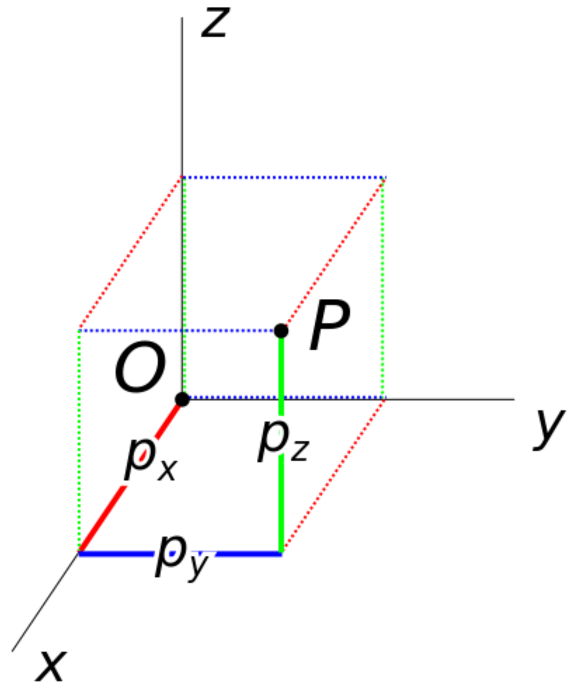

Figure 2: Assigning a vector $\vec{p} \equiv (p_x, p_y, p_z)$ to a point P in space.

Note that, since we have a correspondence between points in space and vectors, we can now talk about *adding* points in space. But what is the geometric meaning then of the point "$P + Q$" obtained by adding a point P (which has a corresponding vector $\vec{p} = (p_x, p_y, p_z)$) to a point Q (which has a corresponding vector $\vec{q}$)? Well according to the rule for vector addition we get a point that we can reach from the origin by:

- traveling in the x-direction a distance $p_x + q_x$, then
- traveling in the y-direction a distance $p_y + q_y$, then
- traveling in the z-direction a distance $p_z + q_z$.

Equivalently, we can obtain the point $P + Q$ by taking the line segment $\overline{OQ}$ and moving it so it starts at P. The point $P + Q$ is then at the end of this displaced line segment.

We can also now *scale* a point P in physical space by a number c. What we end up with is a point cP where the line segment $\overline{OcP}$ (i.e. from O to cP) is collinear with OP but scaled in length by a factor c.

We can also take the *length* of a point P, i.e. $|\vec{p}|$, which you can check is equivalent to the length of the line segment $\overline{OP}$.

Finally, what is the interpretation of the dot product of two points P and Q? This one is far less obvious, but it turns out to be equivalent to taking the product of:

- The length of the line segment $\overline{OP}$, along with
- the length of the line segment $\overline{OQ}$, and also
- the *cosine* of the angle $\angle POQ$,

so that the dot product is a measure of the degree to which the line segments $\overline{OP}$ and $\overline{OQ}$ are *parallel*.