302L S20 Homework 3 solutions

1. The ith sub-segment $\vec{x}_i^- \to \vec{x}_i^+$ starts at a point $\vec{x}_i^- \equiv \vec{x} + (i-1)\frac{\Delta x}{N}\hat{n}$ and ends at $\vec{x}_i^+ \equiv \vec{x} + i\frac{\Delta x}{N}\hat{n}$. If it is not clear why this is then check that:

   - The first subsegment starts at $\vec{x}_1^- = \vec{x}$.
   - The last subsegment ends at $\vec{x}_N^+ = \vec{x} + \Delta x\hat{n}$.
   - $\vec{x}_i^+ = \vec{x}_{i+1}^-$ so that the end of one line subsegment is the beginning of the next.
   - The difference $\vec{x}_i^+ - \vec{x}_i^- = \frac{\Delta x}{N}\hat{n}$ is the same for every i, and when we add up their lengths we get $N\frac{\Delta x}{N} = \Delta x$ which is the length of the full path we want to calculate the work over.

   Therefore our choice of the start and end points $\vec{x}_i^{\pm}$ for the subsegments indeed results in a equal division of the line segment $\vec{x} \to \vec{x} + \Delta x\hat{\mathbf{n}}$ into N equal subsegments.

   We are assuming each subsegment is short enough that the work W_i done over a subsegment can be calculated using the formula for the work W done by a constant force $\vec{F}_o$ over a straight line segment $\vec{x}_1 \to \vec{x}_2$.

$$W = \vec{F}_o \cdot (\vec{x}_2 - \vec{x}_1)$$

   To apply this formula to calculate W_i we make the substitutions:

   - $\vec{x}_1 \to \vec{x}_i^-$
   - $\vec{x}_2 \to \vec{x}_i^+$
   - $\vec{F}_o \to \vec{F}(\vec{x}_i^+)$

   Note that we could equally well substitute in the starting point $\vec{x}_i^-$ into $\vec{F}(\vec{x})$ to get $\vec{F}_o$, or any other point lying on the subsegment[1]. Because the subsegment is so short (i.e. N is so large), the force is essentially constant over the subsegment so it does not matter which point along the subsegment we choose to evaluate $\vec{F}(\vec{x})$ to get $\vec{F}_o$. Choosing the subsegment's endpoint $\vec{x}_i^+$ will end up being simply a more convenient choice than other points.

[1] In fact it would make most sense to use the *midpoint* of the line segment, since it best represents the average value of the force along the subsegment. It turns out for this problem that using the midpoint actually produces an answer that is exactly correct for *any* N, not just N very large.

Plugging in these substitutions we find:

$$W_i = \vec{F}(\vec{x_i^+}) \cdot \left(\vec{x_i^+} - \vec{x_i^-}\right) = -F_N^o \frac{i}{N}\hat{\mathbf{n}} \cdot \frac{\Delta x}{N}\hat{\mathbf{n}} = -F_N^o \Delta x \frac{i}{N^2}$$

2. Summing up all the W_i we get

$$W = \sum_{i=1}^{N} W_i = \sum_{i=1}^{N} \left(-F_N^o \Delta x \frac{i}{N^2}\right) = -F_N^o \Delta x \frac{1}{N^2} \sum_{i=1}^{N} i$$

Here I will walk through how to calculate $\sum_{i=1}^{N}$ for any *even* N:[2]

- The first and last terms in the sum are 1 and N. Adding them together we get $N+1$.
- The second and second-to-last terms are 2 and $N-1$. Adding these together we also $N+1$.
- Continuing in this way we get $N/2$ pairs $(1, N), (2, N-1), \ldots, (N/2, N/2+1)$ that all add to $N+1$. Note this logic only works if N is *even*
- Adding all the pairs together we get $\sum_{i=1}^{N} i = \frac{N}{2}(N+1)$.

so that

$$W = -F_N^o \Delta x \frac{N/2(N+1)}{N^2} = -\frac{1}{2} F_N^o \Delta x \frac{N+1}{N}$$

3. In this problem we are essentially asked to show that

$$\frac{N+1}{N} \approx 1$$

When N is very large. This is somewhat obvious just looking at it (certainly a million plus one is close to a million), but if we want to a more convincing argument we could distribute the denominator to see that

$$\frac{N+1}{N} = \frac{N}{N} + \frac{1}{N} = 1 + \frac{1}{N}$$

As N gets very large, $1/N$ gets very small so that it is very close to zero and we can ignore it[3]. Making this substitution to our expression for the work we get that

$$W = -\frac{1}{2} F_N^o \Delta x$$

Note the work is negative. This makes sense since

[2]The result turns out to be true for odd N as well, and if you're curious I leave it to you to work out why this is.

[3]How close to zero is close enough? Well pick *any* small number you like as your threshold for "close enough" and call it ϵ. I can always find a number N large enough that $1/N$ is closer to zero than your threshold ϵ. This is precisely what mathematicians mean when they say "take the limit where N goes to infinity", i.e.

$$\lim_{N \to \infty} \frac{1}{N} = 0$$

- we expect the confining force to *repel* particles traveling across the surface layer, causing them to slow down and *lose* kinetic energy, and
- by the work-energy principle, the work done along some particle's trajectory is equal to the kinetic energy *gained* by the particle over the course of trajectory.

4. The electric field vector $\vec{E}$ at a point on the surface of a conductor with surface charge density σ is

$$\vec{E} = \frac{\sigma}{\epsilon_o}\hat{n}$$

This electric field vector produces a force $\vec{F}$ on electrons at this point on the surface given by

$$\vec{F} = -e\vec{E} = -\frac{e\sigma}{\epsilon_o}\hat{n}$$

Note this force acts either anti-parallel or parallel to $\hat{n}$ depending on whether σ is positive or negative. We are trying to *eject* electrons, so we want to apply a force parallel (not *anti*parallel) to the surface normal, and therefore we need a negative surface charge density σ.

This is consistent with our conception of particles at the surface of like charge trying to push each other out of the surface. This pushing is typically opposed by the confining force $\vec{F}_N$, which was the subject of the first three questions. In other words, the *net sum* $\vec{F}_\Sigma$ of the forces on an electron at the surface is given by

$$\vec{F}_\Sigma = -e\vec{E} + \vec{F}_N$$

Since these forces must balance in equilibrium (i.e. $\vec{F}_\Sigma = \vec{0}$) we have

$$\vec{F}_N = e\vec{E}$$

This is telling us that as the surface charge density σ (and thus $|\vec{E}|$ increases), the increased mutual repulsion pushes electrons deeper into the surface layer[4], and, as they push deeper, the confining force $|\vec{F}_N|$ they experience increases until equilibrium is reestablished (i.e. force balancing is restored).

At a high enough surface charge density σ the resulting electric field will be stronger than the maximum confining force strength F_N^o. This occurs when

$$-F_N^o\hat{\mathbf{n}} = \frac{e\sigma}{\epsilon_o}\hat{\mathbf{n}}$$

or

$$\sigma = -\frac{F_N^o\epsilon_o}{e}$$

[4] i.e. away from the blue layer and towards the fuschia layer in figure 1 of the homework

I asked you to express the answer not in terms of F_N^o but instead in terms of the work function $\Phi = |W|$, which can be measured using photoemission experiments (discussed in problem 5). Making the appropriate subsitution we get

$$\sigma = -2\frac{\Phi\epsilon_o}{\Delta x e}$$

Note that the vector nature of problem does not really come into play since the vectors fields and line segments involved all point parallel or anti-parallel to one another.

5. (a) After the electron absorbs the photon, it possesses E_γ in kinetic energy. The electron's journey through through the surface layer has an associated work $W = -\Phi$. The work-energy principle states that the change ΔT in a particle's kinetic energy after completing some trajectory is equal to the work W done along that trajectory, i.e.

$$W = \Delta T = T' - T_o$$

where T_o and T' are the initial and final kinetic energies, respectively. Setting $T_o \to E_\gamma$ we have

$$T' = E_\gamma - \Phi$$

(b) In lecture 7 we learned that the potential energy function $U(x, y, z)$ for a particle of charge q between two infinite sheets of opposite surface charge density $\pm\sigma$ is given by

$$U(x, y, z) = -\frac{q\sigma z}{\epsilon_o}$$

where the sheet of surface charge density $+\sigma$ is at $z = 0$ and the sheet of surface charge density $-\sigma$ is at $z = d$. In our current problem we refer to the distance x between the electron and plate A, so in adapting this formula to our problem it is most convenient to let plate A correspond to the sheet of surface charge density $+\sigma$ and make the subsitution of notation $z \to x$. In other words,

$$U(x) = \frac{e\sigma x}{\epsilon_o}$$

is the potential energy of an electron $q = -e$ lying a distance x away from plate A (and thus a distance $d-a$ away from plate B). It remains to determine σ, the surface charge density of plate A. We are given that N_e electrons have been transferred from plate A to plate B. The means there is a net charge $-eN_e$ on plate B and, by charge conservation, a net charge $+eN_e$. Since the plates are, by assumption, conductors, we assume that the charge spreads out evenly over the front faces of the plates. (The "front face" of

plate A is the one facing B and likewise for the front face of B.[5]) Therefore the surface density at the front face of a plate should be the net charge on that plate divided by the area of the face, i.e.

$$\sigma = \frac{eN_e}{\pi r^2}$$

for plate A and $-\sigma$ for plate B. Therefore

$$U(x) = \frac{e^2 N_e x}{\pi r^2 \epsilon_o}$$

(c) Conservation of energy tells us that the sum $E = T(x) + U(x)$ of the electron's kinetic energy and potential energy does not depend on the electron's distance x from plate A. Since we know that at $x = 0$ the particle has a kinetic energy $T(0) = E_\gamma - \Phi$ (part (a)), and a potential energy $U(0) = 0$, then it's kinetic energy $T(d)$ just before reaching plate B is

$$T(d) = T(0) + U(0) - U(d) = E_\gamma - \Phi - \frac{e^2 N_e d}{\pi r^2 \epsilon_o}$$

When $T = 0$, the electron has completely stopped, at which point it turns around and heads back to plate A.[6] Therefore no electron will be able to reach plate B when N_e is so large that

$$T(d) = E_\gamma - \Phi - \frac{e^2 N_e d}{\pi r^2 \epsilon_o} = 0$$

This occurs when

$$N_e^{\text{MAX}} = \frac{\pi r^2 \epsilon_o \left(E_\gamma - \Phi\right)}{e^2 d}$$

Therefore, by measuring the number of electrons N_e transferred between the plates we could obtain a measure of the work function Φ. Actually, if we had a way of measuring the *electric potential* $V(d) = U(d)/(-e) = -\frac{eN_e d}{\epsilon_o}$ between the plates then we could determine Φ by the simple expression

$$\Phi = E_\gamma + eV(d)$$

but more on this later.

[5] If charge accumulated on the back faces of A or B then there would be an electric field *inside* the plates, which is prohibited for conductors. It is also physically reasonable that the opposite charges on the plates would be attracted to each other.

[6] Note that the electron may turn around before $T = 0$ if it has some of its kinetic energy (i.e. velocity) in a direction parallel to the plates. This is familiar from the case of projectile motion, where a projectile may stop rising and begin to fall even when it has still has kinetic energy in the horizontal direction. The process of photoemission is random in the sense that electrons are emitted in a wide range of initial directions so that actually only a small fraction start off traveling straight towards plate B. In any case, the point here is that certainly no electron can continue traveling towards B when *all* of its kinetic energy is gone.