302L S20 Homework 3 - The surface layer and the work function

Due Friday February 14^{th} by 3pm in my mailbox on the fifth floor of PMA (RLM).

Concepts covered: work, electric field at conducting surfaces, field emission, photo-emission, work-energy principle, electric potential between charged plates, conservation of energy.

Note: question 1 starts at the end of page 3

In lecture 6 we argued that a confining force field, which we will call $\vec{F}_N(\vec{x}')$ (in reference to the ***N***ormal direction in which it acts), balances the electrical force $\vec{F}(\vec{x}')$ acting on the charge carriers at the surface of a conductor, so that the charge carriers stay confined to the surface, i.e. so that $\vec{F}(\vec{x}') + \vec{F}_N(\vec{x}') = 0$. The confining force does not originate from the surrounding surface charge density (i.e. it persists even when $\sigma = 0$) but is rather an independent effect arising from the chemistry of the surface.

Let us describe this confining force in the following way. Refer to figure 1. Suppose the surface is not infinitely thin but forms a shell with a very small thickness Δx which we will term the "surface layer". Let $\vec{x}$ be any point on the inner wall of the surface layer (blue on the diagram) and let $\hat{n}$ be the surface normal at the point $\vec{x}$. We define the confining force $\vec{F}_N(\vec{x}')$ for points $\vec{x}'$ in the vicinity of the surface layer as follows:

$$\vec{F}(\vec{x} + x\hat{n}) = \begin{cases} -F_N^o \frac{x}{\Delta x}\hat{n} & 0 < x < \Delta x \\ \vec{0} & x < 0,\, x > \Delta x \end{cases}$$

i.e. the confining force is

- zero everywhere outside the surface layer, and
- linearly increasing from 0 at the inner wall to a maximum strength $F_N^o > 0$ at the outer wall, with a direction pointing in a direction *opposite* the surface normal $\hat{n}$.

The dependence of the strength of the confining force $|\vec{F}_N(x)|$ on the distance x away from the inner wall of the surface layer is plotted in figure 2.

Our conception of the equilibrium distribution of charge at the surface of a conductor is thus the following:

- The charge at a surface occupies an infinitely thin shell composed of all the points lying a constant distance x away from the inner wall of the surface layer.

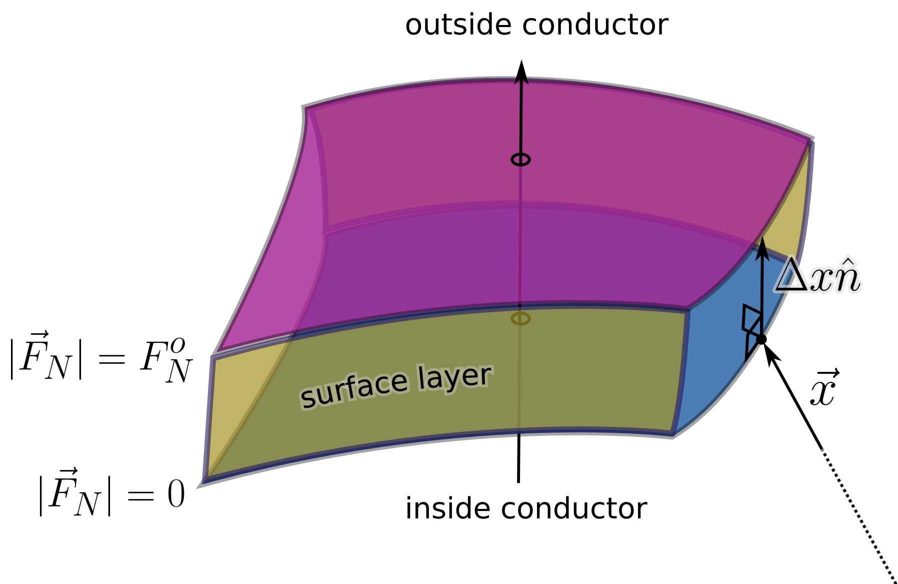

Figure 1: A diagram of some segment of the "surface layer" of a conductor. The inner wall of the surface layer is colored blue and the outer wall is colored fuschia.

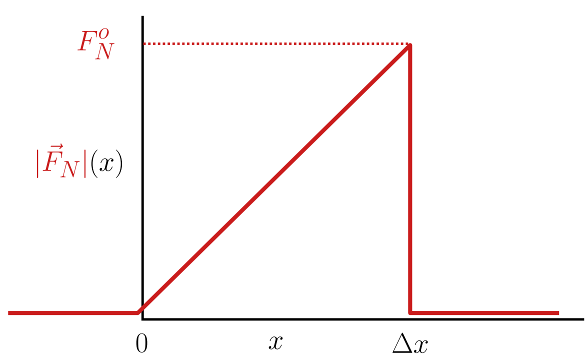

Figure 2: Plot of the confining force strength versus the distance x away from the inner wall.

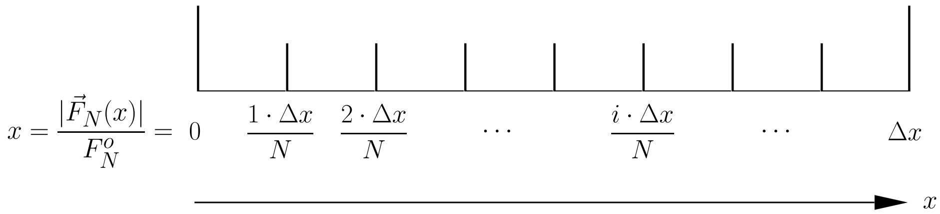

Figure 3: Diagram showing the splitting of the interval $0 < x < \Delta x$ into N even intervals. In the drawing we have $N = 8$.

- This distance x is determined by the requirement of force balancing described earlier: as the surface charge density σ increases, the mutual repulsion of the charge carriers at the surface increases and the charge carriers move deeper into the surface layer (i.e. *x increases*). As the charge moves deeper into the surface layer, the confining force increases in strength until an equilibrium is reached where the forces balance.

Your first task, split across problems 1-3, is to calculate the *work W* required to the take a charge carrier from the inner wall of the surface layer to the outer wall, i.e. from a point $\vec{x}$ to a point $\vec{x} + \Delta x \vec{n}$, or, put more simply, from a distance $x = 0$ from the inner wall to a distance $x = \Delta x$ from the inner wall. Once we obtain the expression for W we move on in problem 4 to determine the surface charge density necessary to overcome the confining force and initiate *field emission*.

Our first task has a complication in that the force field $F_N(\vec{x} + x\hat{n})$ is not constant over the interval $0 < x < \Delta x$ so we can not immediately apply the formula

$$W = \vec{F}_o \cdot (\vec{x}_2 - \vec{x}_1) \tag{1}$$

from lecture 7. Instead we must use the general procedure described in the lecture notes where we split up the interval $0 < x < \Delta x$ into N "sub-intervals" of equal length, where N is large (i.e. the sub-intervals are short)[1]. This division of the interval is illustrated in figure 3.

1. Calculate the work W_i done in taking a charge carrier across the i^{th} interval; that is, from a distance $x = (i-1)\frac{\Delta x}{N}$ away from the inner wall of the surface layer to distance $x = i\frac{\Delta x}{N}$ away. The index i here is any number $1, 2, \ldots, N$.

   Assume N is large enough that the force varies negligibly over any subinterval so that we can use equation (1), taking the constant force $\vec{F}_o$ in the equation to be equal to the value of the force field $\vec{F}(\vec{x})$ evaluated at the subinterval's *end*point. Note that the direction of the force is in the direction opposite to our line segment. How does the dot product in equation (1) simplify?

   The answer should be expressed in terms of F_N^o, Δx, and i.

[1] Be sure not to confuse the N subscript denoting the confining force $\vec{F}(\vec{x}')$ with the number N of subintervals.

2. Now sum up all the works W_i to get the total work $W = \sum_{i=1}^{N} W_i$. You will find your answer contains the sum

$$\sum_{i=1}^{N} i$$

It is actually possible to compute this sum for any N. You can try to Google the answer or use the following trick:

   (i) Assume that N is even. It makes the problem a little easier, and in the end it doesn't matter if N is even or odd so long as it is large.

   (ii) Pair up the first term and last term in the sum. What do they add up to?

   (iii) Now pair up the second term and second-to-last term. What do they add up to?

   (iv) How many such pairs do we have? Remember we assumed that N is even.

   However you decide to find the answer, express it in terms of F_N^o, Δx, and (possibly) N.

3. Now take your result and show that, in the case that N is very large, our result is approximately equal to:

$$W = -\frac{1}{2} F_N^o \Delta x$$

   Notice this result is independent of the number N of subintervals we divided the interval $0 < x < \Delta x$ into. Also notice that the work W is negative, so that, according to the work-energy principle, a particle *loses* kinetic energy in traveling through the surface layer. This is consistent with our conception of the confining force acting to prevent charged particles from escaping the surface layer.

   When we take our conductor to be a metal surface, so that our charge carriers are electrons, the quantity $|W|$ is known as the *work function* of the metal, denoted by the greek letter Φ. It is the minimum kinetic energy an electron must have in order to escape from the metal.

4. Find an expression for the minimum surface charge density σ necessary for overcoming the maximum confining force F_N^o at a metal surface.

   Express your answer in terms of the metal's work function Φ and the thickness Δx of its surface charge layer.

   Hint: How is the electric field at a conducting surface related to the surface charge density at the surface? How does the force on an electron depend on the electric field vector at the electron's location?

5. Two identical circular plates A and B of radius r are separated by a distance $d \ll r$. High energy photons of energy E_γ are focused onto the center of plate A, generating photo-emission of electrons (i.e. "photoelectrons") from the surface of A (see figure 4).

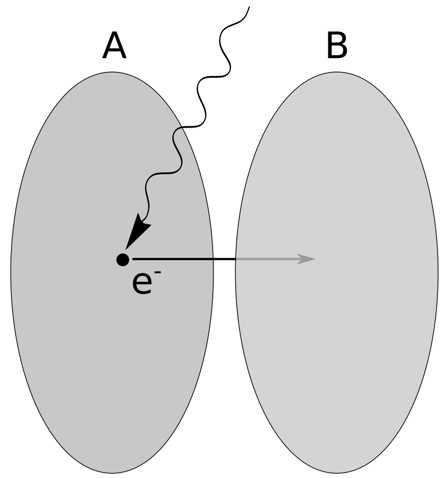

Figure 4: Diagram for question 5. The squiggly line represents a photon, and the straight line represents the emitted electron.

(a) Let $\Phi < E_\gamma$ be the work function for the metal plates. What is the kinetic energy of the photoelectrons immediately outside the surface, i.e. just beyond the surface layer? Assume the electrons have essentially zero kinetic energy before they absorb a photon, and that the plates are uncharged (i.e. $\sigma = 0$).

(b) Suppose the two plates begin initially uncharged, and after an exposure to the high energy photons, we find that N_e photoelectrons have traveled from A and onto B. What is the electric potential difference $V(x)$ at a distance x away from plate A after the exposure? Assume $V(0) = 0$. Because the plates are spaced close together, we can assume each generates a constant electric field given by the formula for an *infinite* charge plate.

(c) Now suppose we indefinitely expose plate A to photons. The transfer of electrons from $A \to B$ is found to cease after some time. Why? How many total electrons N_e^{MAX} can the photons transfer from $A \to B$ before the transfer ceases? Assume that, no matter how large N_e gets, the electric field at the surface of neither A nor B is strong enough to overcome the max confinement force F_N^o.

Hint: what electrical forces will a photon electron traveling between A and B experience when N_e is large? Can we analyze this situation in terms of energy conservation, i.e. use the results of the previous problem?