HW #2: vector practice

due Friday, Feb. 7th before class

***Note:** The purpose of this exercise is to build up your confidence with vectors. Using a small number of definitions and the familiar properties of ordinary numbers (e.g.* $1.5, \pi, -2.45, \frac{1}{3}$, *and so on) we can prove to ourselves many things about vectors. As you go about doing these "mini-proofs", only use the definitions I've provided in the document, along with what you already know about ordinary numbers. Apply only one logical step at a time, so that you can clearly see where you are simply applying the definitions I give and where you are applying what you already know about ordinary numbers. At the end I've provided solutions to some of the problems that you can use as a guide for the others.*

Organize your work so that your reasoning is easy to follow. Do not write too small and use lots of whitespace for readability. Use my solutions at the bottom of the document as an example, though it is not necessary to write so much english so long as one step clearly follows from the previous.

For our purposes, a *vector* will be defined as an ordered sequence of numbers. That's all a vector is! Let the simplicity of the definition comfort you and provide you with security – whenever we talk about vectors we are, in the end, talking simply about a sequence of numbers. For problems dealing with three-dimensional space we have a sequence of three numbers (also known as a *3-tuple*), and for problems dealing with two dimensional space we have a sequence of two numbers (aka a *2-tuple*). In the following I present formulas that apply for the three-dimensional case, but the necessary modifications for the two-dimensional case are straightforward[1].

We can conveniently specify vectors using the $(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ notation, so that the first of our three numbers goes in the first slot, the second number goes in the second slot, and the third number in the third slot.

So, for instance, one vector could be the following:

$$(+1.5, -1.5, +3.5)$$

What about, say, the symbol $\vec{v}$? Isn't that a vector too? Well, unless there has been some rule specified for how this symbol $\vec{v}$ corresponds to an ordered sequence of three numbers, than we can not say whether or not it is a vector. Oftentimes though we say "let $\vec{v}$ be some vector". By this we mean, "Pick any sequence of three numbers, however you wish. The following argument will not depend on your exact selection." Let's call our selection (v_x, v_y, v_z). Ok, then, **now** $\vec{v} = (v_x, v_y, v_z)$ represents an honest-to-god vector as we've defined them in the first paragraph.

Our specification of vectors is, however, not yet complete. Vectors are more specialized than just sequences of three numbers. For instance, for vectors we have the operation of *scalar multiplication*, where we can take a *number* c and a *vector* $\vec{v} = (v_x, v_y, v_z)$ and obtain another vector, denoted $c\vec{v}$. Note yet that our definition of scalar multiplication is not yet complete because haven't said what sequence of three

[1]See Question 4 for a problem dealing with the two-dimensional case.

numbers corresponds to $c\vec{v}$. The sequence is given by the following simple rule:

$$c\vec{v} = (cv_x, cv_y, cv_z)$$

Further, we have the operation of *vector addition*, where we take two *vectors* $\vec{v} = (v_x, v_y, v_z)$ and $\vec{w} = (w_x, w_y, w_z)$ and obtain another *vector* $\vec{v} + \vec{w}$, given by the sequence

$$\vec{v} + \vec{w} = (v_x + w_x, v_y + w_y, v_z + w_z)$$

Question 1:

a) Exploit the *commutativity of addition* for ordinary numbers, i.e. the fact that for any two numbers a and b we have that

$$a + b = b + a$$

to show that vector addition also commutes, i.e. that for any two vectors $\vec{v}$ and $\vec{w}$ we have that:

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

b) Now exploit the *distributive property of multiplication* for ordinary numbers, i.e. the fact that for any three numbers a, b, and c we have that

$$a(b + c) = ab + bc$$

to show that, for any number c and any two vectors $\vec{v}$ and $\vec{w}$ we have that

$$c\,(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$$

Our vectors have one more operation, a *dot product*, that takes two *vectors* $\vec{v} = (v_x, v_y, v_z)$ and $\vec{w} = (w_x, w_y, w_z)$ and returns a *number*, which is given by the rule

$$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z$$

Question 2:

a) Use the *commutativity of multiplication* for ordinary numbers (i.e. for any two numbers a and b we have $ab = ba$) and a line of reasoning similar to the one I used about for vector multiplication to show that the dot product is *symmetric* in the sense that:

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$$

b) Use the distributive property of multiplication of ordinary numbers (explained earlier) to show that the dot product also distributes in the sense that for any three vector $\vec{v}$, $\vec{u}$, and $\vec{w}$ we have:

$$\vec{v} \cdot (\vec{u} + \vec{w}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w}$$

c) Use the distributive property again, this time “in reverse”, to show that for any two vectors $\vec{v}$, $\vec{w}$ and any number c we have:

$$(c\vec{v}) \cdot \vec{w} = c\,(\vec{v} \cdot \vec{w})$$

as well as (via the symmetric relation proved in part a)

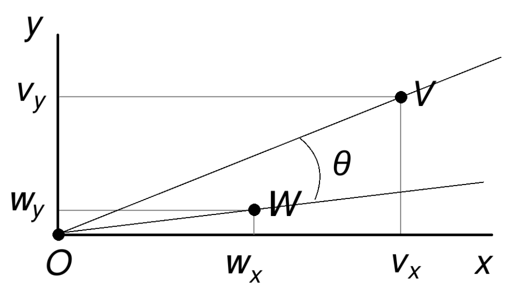
$$\vec{v} \cdot (c\vec{w}) = c\,(\vec{v} \cdot \vec{w})$$

From the dot product we can also define the *length* of a vector $\vec{v}$, denoted by $|\vec{v}|$, using the following rule:

$$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} \tag{1}$$

Question 3: Show that the dot product $\vec{v} \cdot \vec{v}$ of a vector $\vec{v}$ with itself never *negative* (so that there is never any issue in taking a square root to obtain the vector's length).

Question 4:



Let $V \equiv \vec{v} = (v_x, v_y)$ and $W \equiv \vec{w} = (w_x, w_y)$ be two points on a plane, where

- v_x is the x coordinate of the point V
- v_y is the y coordinate of the point V
- w_x is the x coordinate of the point W
- w_y is the y coordinate of the point W

a) Show that the length of the vector $\vec{v} - \vec{w}$ as we've defined it in equation (1) is consistent with the familiar expression given by the Pythagorean theorem for the length of the line segment WV.

b) Use the law of cosines to prove the useful relationship

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$$

where θ is the angle WOV and the point O is the origin $(0,0) \equiv \vec{0}$. What is the dot product of two vectors that are parallel? Anti-parallel? Perpendicular?

Let us now define three special vectors:

- $\hat{\mathbf{i}} \equiv (1, 0, 0)$
- $\hat{\mathbf{j}} \equiv (0, 1, 0)$
- $\hat{\mathbf{k}} \equiv (0, 0, 1)$

together these vectors are known as the *standard basis vectors*.

Question 5:

a) Show that any vector $\vec{v} = (v_x, v_y, v_z)$ can be written as a *linear combination* of the three standard basis vectors, i.e.

$$\vec{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$$

In other words we say that the vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ "span the space of vectors". When we also take into account that the linear combination $v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ is *unique*, we can say that the vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ comprise a *basis* for our vectors.

b) Show that the standard basis vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are *unit vectors* (or "normalized") in the sense that:

$$|\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1$$

c) Finally show that they *mutually orthogonal* in the sense that:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$$

A basis of mutually orthogonal unit vectors is known as an *ortho-normal basis.*

Selected answers
Q1a:

Recall that by a vector $\vec{v}$ is implicitly understood to mean some sequence of three numbers, say (v_x, v_y, v_z). Likewise, $\vec{w} \leftrightarrow (w_x, w_y, w_z)$. Beginning with the *definition* of vector addition we have

$$\vec{v} + \vec{w} = (v_x + w_x, v_y + w_y, v_z + w_z)$$

then, exploiting the commutativity of ordinary numbers, we *flip* the order of addition for each of the three numbers in the sequence, i.e.

$$(v_x + w_x, v_y + w_y, v_z + w_z) = (w_x + v_x, w_y + v_y, w_z + v_z)$$

By inspection, the term on the right hand side is precisely the *definition* of the expression $\vec{w} + \vec{v}$, i.e.

$$(w_x + v_x, w_y + v_y, w_z + v_z) = \vec{w} + \vec{v}$$

and so our work is done, since we have joined the left hand side of the original "theorem" $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ to the right hand by a series of expressions joined together by equals signs (=). As they say in the biz, Q.E.D.!

Q2c:

Applying the definition of the dot product we have

$$(c\vec{v}) \cdot \vec{w} = cv_x w_x + cv_y w_y + cv_z w_z$$

Applying the distributive property of multiplication "in reverse", we see that

$$cv_x w_x + cv_y w_y + cv_z w_z = c\,(v_x w_x + v_y w_y + v_z w_z)$$

the term in the parentheses on the right hand side can be seen to be precisely the *definition* of the dot product $\vec{v} \cdot \vec{w}$, i.e.

$$c\,(\vec{v} \cdot \vec{w})$$

Q.E.D.
To prove the second relation, we first exploit the symmetry of the dot product to assert

$$\vec{v} \cdot (c\vec{w}) = (c\vec{w}) \cdot \vec{v}$$

then we use the result from the part a) and find that

$$(c\vec{w}) \cdot \vec{v} = c\,(\vec{w} \cdot \vec{v})$$

and finally use the symmetry property again to get

$$c\,(\vec{w} \cdot \vec{v}) = c\,(\vec{v} \cdot \vec{w})$$

Q.E.D.
Q4b:
This problem is difficult. Make a triangle from the vertices O, W, V. The law of cosines says that the length of the line segment WV is related to the length of the line segments OV and OW and the angle $WOV \equiv \theta$ by the following equation

$$(WV)^2 = (OV)^2 + (OW)^2 - 2(OW)(OV)\cos\theta$$

From the previous problem, however, we now know that

$$WV = |\vec{v} - \vec{w}|$$

and, by the same token,

$$OV = |\vec{v} - \vec{0}| = |\vec{v}|$$

and likewise

$$OW = |\vec{w} - \vec{0}| = |\vec{w}|$$

By squaring both sides of the definiton of the vector length, i.e.

$$|\vec{v}| \equiv \sqrt{\vec{v} \cdot \vec{v}} \longrightarrow |\vec{v}|^2 = \vec{v} \cdot \vec{v}$$

we find we can rewrite the previous three results as

$$(WV)^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})$$

and likewise

$$(OV)^2 = \vec{v} \cdot \vec{v}$$

and finally

$$(OW)^2 = \vec{w} \cdot \vec{w}$$

The final piece we need is to make use of the distributive property of the dot product to obtain

$$(WV)^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{w} - \vec{w} \cdot \vec{v} + \vec{w} \cdot \vec{w}$$

this can be simplified after applying the symmetry property of the dot product to obtain

$$\vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{w} - \vec{w} \cdot \vec{v} + \vec{w} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} - 2\vec{v} \cdot \vec{w}$$

Making a bunch of substitutions into both sides of the first equation we arrive at

$$\vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} - 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} - 2|\vec{v}||\vec{w}| \cos\theta$$

cancelling identical terms on the left and right hand sides we get

$$-2\vec{v} \cdot \vec{w} = -2|\vec{v}||\vec{w}| \cos\theta$$

i.e.

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$$

Q.E.D.

When two vectors are parallel we mean their mutual angle is $0°$, so that

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos 0° = |\vec{v}||\vec{w}|$$

while anti-parallel implies a mutual angle of $180°$, i.e.

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos 180° = -|\vec{v}||\vec{w}|$$

and, finally, perpendicular implies a mutual angle of $90°$, resulting in

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos 90° = 0$$